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The Virtual Element Method for an Elliptic Hemivariational Inequality with Convex Constraint

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Abstract. An abstract framework of numerical method is devised for solving an elliptic hemivariational inequality with convex constraint. Convergence of the method is explored under the minimal solution regularity available from the well-posedness of the hemivariational inequality. A Céa-type inequality is derived for error estimation. As a typical example, a virtual element method is proposed to solve a frictionless unilateral contact problem and its optimal error estimates are obtained as well. Numerical results are reported to show the performance of the proposed method.

AMS subject classifications: 65N30

Key words: Virtual element method, hemivariational inequality, error estimate, multiobjective double bundle method.

1. Introduction

Hemivariational inequalities (HVIs) arise in the study of various industrial processes and engineering applications. In the past three decades, many researchers have contributed mathematical theories for such models (cf. [18, 32, 37, 39–41]). The finite element method has been used to solve them with systematic theoretical analysis (cf. [6, 27, 29–32]). We refer the reader to the survey paper [28] for details along this line. In this paper, we intend to propose and analyze the virtual element method for an elliptic hemivariational inequality with convex constraint, which can be viewed as an extension of our earlier work in [23].

We first introduce some notation about function spaces for later uses. Throughout

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this paper, we will use the standard notation for Sobolev spaces and their norms and seminorms (cf. [1]). Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open bounded domain with Lipschitz boundary $\partial\Omega$. Given an integer $m \geq 1$, let X be a closed subspace of $H^1(\Omega; \mathbb{R}^m)$ and K a closed and convex subset of X with $0_X \in K$. Moreover, let X_j be another Banach space and $\gamma_j \in \mathcal{L}(X, X_j)$. Then, the mathematical problem to be studied can be described as follows.

Problem (P). Find an element $u \in K$ such that

$$a(u, v-u) + j^0(\gamma_j u; \gamma_j v - \gamma_j u) \geq \langle f, v-u \rangle, \quad \forall v \in K, \tag{1.1}$$

where $a(\cdot,\cdot)$ is a bilinear form over $X, j : X_j \to \mathbb{R}$ is a locally Lipschitz function, f is a bounded linear functional over X, while $j^0(x; v)$ denotes the generalized Clarke directional derivative of j at x in a direction v defined by (cf. [21])

$$j^0(x; v) = \limsup_{y \to x,\, \lambda \downarrow 0} \frac{1}{\lambda}\big(j(y + \lambda v) - j(y)\big).$$

Recently, virtual element methods (VEMs) were developed and have gained popularity as a numerical approach for solving partial differential equations (PDEs), started with [2, 7, 9]. VEMs have some advantages over standard finite element methods. For example, they are more convenient to handle PDEs on complex geometric domains or the ones associated with high-regularity admissible spaces. The methods have been applied to solve many different kinds of mathematical physical problems, e.g., conforming and nonconforming VEMs for second-order elliptic problems [5, 17, 24, 35], fourth-order problems [3,16,44], elasticity problems [8,45], and $(2m)$-th-order elliptic problems in any dimensions (cf. [20]). Some systematic theoretical analyses were given in [10, 14, 19, 20] for conforming and nonconforming VEMs. In the reference [23], we introduced an abstract framework of numerical method and established an error analysis for the problem (1.1) without constraint, i.e. for the case $K = X$. We applied the VEM for solving two contact problems and derived optimal order error estimates of their numerical solutions under appropriate solution regularity assumptions.

In this paper, we first extend the ideas in [23] to devise naturally an abstract framework of numerical method for the problem (1.1). Then, we extend arguments presented in [25, 31] in a subtle way to show convergence of the numerical solutions. In addition, we derive a Céa-type inequality by using the techniques in [23,25]. As a typical example, we apply the previous results to propose a VEM for a frictionless unilateral contact problem and derive its optimal error estimates. Under some assumptions, this discrete problem is equivalent to a minimization problem (cf. [26, 32]), from which we are able to compute the numerical solution by means of the multiobjective double bundle method (cf. [38]) effectively. Finally, we provide several numerical results to show the performance of the proposed method.

We end this section by introducing some basic results for later requirements. The following elementary inequality is simple but useful in the forthcoming analysis:

$$a, b, x \geq 0 \quad \text{and} \quad x^2 \leq ax + b \quad \Longrightarrow \quad x^2 \leq a^2 + 2b. \tag{1.2}$$