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A Parameterized Class of Complex Nonsymmetric Algebraic Riccati Equations

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Abstract. In this paper, by introducing a definition of parameterized comparison matrix of a given complex square matrix, the solvability of a parameterized class of complex nonsymmetric algebraic Riccati equations (NAREs) is discussed. The existence and uniqueness of the extremal solutions of the NAREs is proved. Some classical numerical methods can be applied to compute the extremal solutions of the NAREs, mainly including the Schur method, the basic fixed-point iterative methods, Newton's method and the doubling algorithms. Furthermore, the linear convergence of the basic fixed-point iterative methods and the quadratic convergence of Newton's method and the doubling algorithms are also shown. Moreover, some concrete parameter selection strategies in complex number field for the doubling algorithms are also given. Numerical experiments demonstrate that our numerical methods are effective.

AMS subject classifications: 65F15, 65F18, 65F30

Key words: Complex nonsymmetric algebraic Riccati equation, extremal solution, numerical method, doubling algorithm, complex parameter selection strategy.

1. Introduction

A complex nonsymmetric algebraic Riccati equation (NARE) has the form

$$XCX - XD - AX + B = 0, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{m\times n}, C \in \mathbb{C}^{n\times m}$ and $D \in \mathbb{C}^{n\times n}$ are four known matrices and $X \in \mathbb{C}^{m\times n}$ is a matrix to be found.

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Let

$$\mathcal{H} = \begin{pmatrix} D & -C \\ B & -A \end{pmatrix}, \quad \mathcal{Q} = \mathcal{J}\mathcal{H} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}, \tag{1.2}$$

where $\mathcal{J} = \begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$. The complementary (dual) nonsymmetric algebraic Riccati equation of (1.1) is

$$YBY - YA - DY + C = 0, \tag{1.3}$$

which will be abbreviated as cNARE (dNARE), where $Y \in \mathbb{C}^{n\times m}$ is a matrix to be found.

Complex nonsymmetric algebraic Riccati equations (NAREs) arise in many applied fields, such as the transient analysis of Markov modulated fluid flow model [23], open-loop Nash equilibrium of linear quadratic differential games [13], applied probability [11] and transport theory [15], and other applications [20, 24]. Here we consider a Markov modulated fluid flow $\{(F(t), J(t)) : t \geq 0\}$, where $F(t)$ and $J(t)$ denote the fluid level in a container and the state of a Markov chain modulating the fluid process at time t, respectively. The finite state space of the Markov chain can be partitioned as $S = S_1 \cup S_2 \cup S_3$, where the cardinality of S_i is n_i, $i = 1, 2, 3$, and its generator T can be partitioned as

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix},$$

where $[T]_{ij} \geq 0$ for $i \neq j$, $[T]_{ii} = -\sum_{j\neq i}[T]_{ij}$ and the size of T_{ij} is $n_i \times n_j$. Given positive $r_i, i \in S_1 \cup S_2$, suppose that the fluid level increases at rate r_i if the chain is in state $i \in S_1$, the level decreases at rate r_i provided it is positive if in state $i \in S_2$ and the level is a constant if in state $i \in S_3$. Let $\tau = \inf\{t > 0 : F(t) = 0\}$. For $i \in S_1, j \in S_2$, define

$$\begin{aligned} &P_{ij}(u) = P\big[\tau \leq u, J(\tau) = j | F(0) = 0, J(0) = i\big], \\ &[\Phi(\xi)]_{ij} = \int_0^\infty e^{-\xi u} dP_{ij}(u). \end{aligned}$$

The joint distribution of fluid level and state can be characterized by $\Phi(\xi)$. If $\xi I - T_{33}$ is invertible, then let

$$T(\xi) = \begin{pmatrix} T_{11}(\xi) & T_{12}(\xi) \\ T_{21}(\xi) & T_{22}(\xi) \end{pmatrix}$$

be the Schur complements of $\xi I - T_{33}$ in $\xi I - T$, and

$$\begin{aligned} Q_{11}(\xi) = C_1^{-1}T_{11}(\xi), \quad Q_{12}(\xi) = C_1^{-1}T_{12}(\xi), \\ Q_{21}(\xi) = C_2^{-1}T_{21}(\xi), \quad Q_{22}(\xi) = C_2^{-1}T_{22}(\xi), \end{aligned}$$

where C_1 and C_2 are two diagonal matrices with diagonal elements $r_i, i \in S_1$ and $r_i, i \in S_2$, respectively. The analysis in [1, 3, 4] has shown that $\Phi(\xi)$ is a solution of

$$XQ_{21}(\xi)X - XQ_{22}(\xi) - Q_{11}(\xi)X + Q_{12}(\xi) = 0. \tag{1.4}$$