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Critical Quenching Exponents for Heat Equations Coupled with Nonlinear Boundary Flux*

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Abstract: We discuss the quenching phenomena for a system of heat equations coupled with nonlinear boundary flux. We determine a critical value for the exponents in the boundary flux, such that only in the super critical case the simultaneous quenching can happen for any solution.

Key words: quenching, simultaneous, non-simultaneous, nonlinear boundary flux

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1 Introduction

This paper is devoted to discussing the quenching phenomena for the following parabolic system with the nonlinear boundary flux of negative exponents

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, \quad \dfrac{\partial v}{\partial t} = \Delta v, & \text{for } (x,t) \in B \times (0,T), \\ \dfrac{\partial u}{\partial \eta} = -v^{-p}, \quad \dfrac{\partial v}{\partial \eta} = -u^{-q}, & \text{for } (x,t) \in \partial B \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \text{for } x \in B, \end{cases} \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^n$, η is the unit outward normal to ∂B, $p, q > 0$, and $u_0(x)$, $v_0(x)$ are radially symmetric, positive, smooth and satisfy some suitable compatibility conditions on the boundary.

Quenching phenomena has been studied by many authors for a variety of problems (see for instance [1]–[10] and the references therein). In [1]–[3], for the one-dimensional case, the quenching phenomena for the system of heat equations with coupled nonlinear boundary sources and nonlinear inner sources have been studied respectively. Many works are devoted to investigating the quenching phenomena for a single equation (see [4]–[10]). In terms of a system of equations, simultaneous or non-simultaneous quenching phenomena, as far as we know, has been referred by only a few authors.

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The purpose of the present paper is to study the critical quenching exponents of the problem (1.1). This is motivated by the recent work [1], in which the authors investigated quite an interesting simultaneous and non-simultaneous quenching phenomena under some convexity assumptions on the initial data. Roughly speaking, for the simultaneous quenching phenomenon, we mean that for some time $T > 0$, each component of the solution (u, v) vanishes as $t \to T^-$, while the time derivatives blow-up at the same time. For the non-simultaneous quenching phenomenon, we mean that only one component vanishes. What we want to know is that, for fixed exponents p and q, whether the simultaneous quenching happens for all solutions with any initial datum. Precisely speaking, we are interested in seeking a subset $Q \subset \mathbb{R}_+ \times \mathbb{R}_+$, such that for any fixed $(p, q) \in Q$, the simultaneous quenching phenomenon happens for any solution (u, v), while for any fixed $(p, q) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus Q$, non-simultaneous quenching phenomenon happens for at least one solution (u, v). In fact, it shows that

$$Q = \{(p, q);\ p \geq 1,\ q \geq 1\}.$$

In other words, unconditionally simultaneous quenching phenomenon happens for all solutions if and only if $p \geq 1$ and $q \geq 1$, namely, $p_c = 1$, $q_c = 1$ are the critical values of the exponents p and q. It should be noticed that, to establish such a result, the method used in the previous works could not be directly applied to our problem, since we must remove the convexity assumptions on the initial data.

This paper is organized as follows. In Section 2, we present our main result and give some auxiliary lemmas. The proof will be divided into several propositions in the subsequent section.

2 The Main Result and Auxiliary Lemmas

Let u_0, v_0 be radially symmetric. Then the corresponding radial problem for the original problem (1.1) can be given by the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{n-1}{r}\dfrac{\partial u}{\partial r}, \quad \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{n-1}{r}\dfrac{\partial v}{\partial r}, & \text{for } (r,t) \in (0,1) \times (0,T), \\ \dfrac{\partial u}{\partial r}(1,t) = -v^{-p}(1,t), \quad \dfrac{\partial v}{\partial r}(1,t) = -u^{-q}(1,t), & \text{for } t \in (0,T), \\ \dfrac{\partial u}{\partial r}(0,t) = 0, \quad \dfrac{\partial v}{\partial r}(0,t) = 0, & \text{for } t \in (0,T), \\ u(r,0) = u_0(r), \quad v(r,0) = v_0(r), & \text{for } r \in (0,1). \end{cases} \tag{2.1}$$

Here, for the sake of the simplicity of notations, we still use $(u(r, t), v(r, t))$ to denote a solution, although it is a function with two variables while a solution of (1.1) is a function with $n + 1$ variables.

The main result of this paper is the following theorem.

Theorem 2.1 *If $p, q \geq 1$, then simultaneous quenching must happen for any solution of the problem* (2.1). *Otherwise, if* $\min\{p, q\} < 1$, *then for every such* (p, q), *there exists at*