COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
25(2)(2009), 159–164

A Theorem of Nehari Type on the Bergman Space*

LU YU-FENG AND YANG JUN
(*Department of Applied Mathematics, Dalian University of Technology, Dalian, 116024*)

Communicated by Ji You-qing

Abstract: In this paper we concern with the characterization of bounded linear operators S acting on the weighted Bergman spaces on the unit ball. It is shown that, if S satisfies the commutation relation $ST_{z_i} = T_{\bar{z}_i}S(i = 1, \cdots, n)$, where $T_{z_i} = z_i f$ and $T_{\bar{z}_i} = P(\bar{z}_i f)$ where P is the weighted Bergman projection, then S must be a Hankel operator.

Key words: Bergman space, unit ball, Hankel operator
2000 MR subject classification: 47B35, 47B47
Document code: A
Article ID: 1674-5647(2009)02-0159-06

1 Introduction

Let $\mathbb{C}$ denote the set of complex numbers. Throughout this paper we fix a positive integer n. Let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. Addition, scalar multiplication, and conjugation are defined on $\mathbb{C}^n$ componentwise. For $z = (z_1, \cdots, z_n)$, $w = (w_1, \cdots, w_n) \in \mathbb{C}^n$, we define

$$\langle z,\ w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n,$$

where $\bar{w}_k$ is the complex conjugate of w_k. For a multi-index $m = (m_1,\ \cdots,\ m_n)$ and $z = (z_1,\ \cdots,\ z_n) \in \mathbb{C}^n$, we also write

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}, \qquad z^m = z_1^{m_1} \cdots z_n^{m_n}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$B_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

We let dV denote the volume measure on B_n, normalized so that $V(B_n) = 1$. For $\alpha > -1$, the weighted Lebesgue measure dV_α is defined by

$$dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dV(z),$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}$$

***Received date:** Jan. 26, 2005.
Foundation item: The NSF (10671028) of China.

is a normalizing constant so that dV_α is a probability measure on B_n.

For $p \geq 1$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of holomorphic functions f in $L^p(B_n,\ dV_\alpha)$, that is,

$$A^p_\alpha = L^p(B_n,\ dV_\alpha) \cap H(B_n).$$

When the weight $\alpha = 0$, for the sake of simplicity, we write A^p to denote A^p_α. These are the standard (unweighted) Bergman sapces.

The Bergman space A^p_α is a closed subspace of $L^p(B_n,\ dV_\alpha)$ and the set of polynomials is dense in A^p_α (see [1]).

For $\phi \in L^\infty(B_n)$, the Hankel operator H_ϕ is defined on A^2_α by

$$H_\phi(f) = P(J(\phi f)),$$

where J is the unitary operator defined on $L^2(B_n,\ dV_\alpha)$ by

$$J(f(z)) = J(f(z_1,\ \cdots,\ z_n)) = f(\bar{z}) = f(\bar{z}_1,\ \cdots,\ \bar{z}_n)$$

and P is the weighted Bergman projection from $L^2(B_n,\ dV_\alpha)$ onto A^2_α.

The Toeplitz operator with the symbol $\phi \in L^\infty(B_n)$ is defined on A^2_α by

$$T_\phi f = P(f\phi), \qquad f \in A^2_\alpha.$$

The symbol z_i denote the i-th coordinate function $(i = 1, \cdots, n)$.

It is easily established that

$$H_\phi T_{z_i} = T_{\bar{z}_i} H_\phi.$$

Thus, the Hankel operators H_ϕ are special instances of solutions of the operator equation

$$ST_{z_i} = T_{\bar{z}_i} S \qquad (i = 1, \cdots, n), \tag{1.1}$$

where S is a bounded linear operator on A^2_α.

It is well known that on the classic Hardy space H^2, Toeplitz operators and Hankel operators are of the same status, and present different operators classes. Halmos[2] regarded Hankel operators as essential parts of the Toeplitz operator theory, and many authors studied Hankel operators and their related problems in [2], [3], [4] and [5].

On the Hardy space H^2, Nehari[6] proved that if S is a bounded linear operator such that

$$ST_z = T_{\bar{z}} S,$$

then $S = H_\phi$ for some $\phi \in L^\infty$. Moreover, ϕ can be chosen such that

$$\|H_\phi\| = \|\phi\|.$$

Faour[7] proved the theorem of Nerhari Type on Bergman spaces of the unit disk. The first author and Sun[8] gave the characterization of Hankel operators on the generalized H^2 spaces, which is also similar to the Nehari theorem of Hardy space.

The motivation of this paper is the question whether solutions of the operator equation (1.1) must be the Hankel operator on the Bergman space A^2_α.

In this paper we take the weighted Bergman space A^2_α as our domain and prove a Nerhari Type theorem. While our method is basically adapted from [7] and [8], substantial amount of extra work is necessary for the setting of the weighted Bergman spaces on the unit ball.