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Energy Decay of Solutions to a Nondegenerate Wave Equation with a Fractional Boundary Control

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Abstract. In this paper, we study the energy decay rate for a one-dimensional nondegenerate wave equation under a fractional control applied at the boundary. We proved the polynomial decay result with an estimation of the decay rates. Our result is established using the frequency-domain method and Borichev-Tomilov theorem.

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1 Introduction

In this paper, we are concerned with the boundary stabilization of convolution type for nondegenerate wave equation of the form

$$w_{tt}(x,t)-(a(x)w_x(x,t))_x=0 \text{ in } (0,1)\times(0,\infty), \tag{1.1}$$

where the coefficient a is a positive function on $[0,1]$.

Up to now, there are many works concerning the stabilization and controllability of nondegenerate wave equation with different types of dampings (see e.g. [1–4] and the references therein). In [4], for $a(x)=a_1x+a_0$: the authors have established asymptotics stabilization under boundary conditions of the form

$$\begin{cases} (aw_x)(0,t)=0, \\ (aw_x)(1,t)=-kw(1,t)-w_t(1,t), \quad k>0. \end{cases}$$

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It has been shown in [1], for $a\in H^1(0,1), a(x)\geq a_0>0$, that the feedback law

$$\begin{cases}(aw_x)(0,t)=-cw(0,t)-F(w_t(0,t), \quad c>0,\\ Mw_{tt}(1,t)+(aw_x)(1,t)=0.\end{cases}$$

exponentially stabilizes equation (1.1) under appropriate assumptions on the function F. Another stabilization result for equation (1.1) has also been established in [3] via the action of the following feedback:

$$\begin{cases}(aw_x)(0,t)=-cw(0,t)-F(w_t(0,t),\\ (aw_x)(1,t)=-cw(1,t)-F(w_t(1,t), \quad c>0.\end{cases}$$

In [2], the authors considered the following modelization of a flexible torque arm controlled by two feedbacks depending only on the boundary velocities:

$$\begin{cases}w_{tt}(x,t)-(a(x)w_x)_x+\alpha w_t(x,t)+\beta w(x,t)=0, \quad 0<x<1,\ t>0,\\ (a(x)w_x)(0)=k_1w_t(0,t), \quad t>0,\\ (a(x)w_x)(1)=-k_2w_t(1,t), \quad t>0,\end{cases}$$

where

$$\begin{cases}\alpha\geq 0,\beta>0,\ k_1,k_2\geq 0,\ k_1+k_2\neq 0,\\ a\in W^{1,\infty}(0,1),\ a(x)\geq a_0>0 \ \text{ for all } \ x\in[0,1].\end{cases}$$

They proved the exponential decay of the solutions.

Motivated by the work of [2], a feedback control depending only on the velocity has been proposed in [5] for the system (1.1) and an asymptotic convergence result has been established (see also [6–8]).

In this article, we are concerned with the system

$$\begin{cases}w_{tt}(x,t)-(a(x)w_x(x,t))_x=0, & \text{in } (0,1)\times(0,+\infty),\\ w(0,t)=0, & \text{on } (0,+\infty),\\ (aw_x)(1,t)=-\varrho\partial_t^{\alpha,\eta}w(1,t), & \text{on } (0,+\infty),\\ w(x,0)=w_0(x),w_t(x,0)=w_1(x), & \text{on } (0,1),\end{cases} \tag{P}$$

where $\varrho>0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative (see [9] and [10]) defined by the following formula:

$$\partial_t^{\alpha,\eta}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}e^{-\eta(t-s)}\frac{\mathrm{d}f}{\mathrm{d}s}(s)\mathrm{d}s, \quad \eta\geq 0,$$

where Γ is the usual Euler gamma function and $(0<\alpha<1)$.

Although there is quite a bit of work on damping mechanisms for beam models of this kind, there does not seem to be much about damping involving fractional derivatives. In [11], Mbodje studies the energy decay of the wave equation (with $a\equiv 1$) with a