COMMUNICATIONS IN MATHEMATICAL RESEARCH
25(3)(2009), 193–203

Finding Periodic Solutions of High Order Duffing Equations via Homotopy Method*

YANG XUE AND XU XU
(*College of Mathematics, Jilin University, Changchun, 130012*)

Communicated by Li Yong

Abstract: This paper presents a detailed analysis of finding the periodic solutions for the high order Duffing equation

$$x^{(2n)} + g(x) = e(t) \qquad (n \geq 1).$$

Firstly, we give a constructive proof for the existence of periodic solutions via the homotopy method. Then we establish an efficient and global convergence method to find periodic solutions numerically.

Key words: high order Duffing equation, periodic solution, homotopy method
2000 MR subject classification: 65M02, 65N15, 65R43
Document code: A
Article ID: 1674-5647(2009)03-0193-11

1 Introduction

In this paper, we concerned with the problem of finding the periodic solutions for the following high order Duffing equation

$$x^{(2n)} + g(x) = e(t), \quad x \in \mathbf{R}, \tag{1.1}$$

where $g : \mathbf{R} \to \mathbf{R}$ is a C^1 function, $e : \mathbf{R} \to \mathbf{R}$ is a continuous function, and

$$e(t) = e(t + 2\pi), \qquad t \in \mathbf{R}.$$

The periodic solutions for the second order Duffing equation have been widely investigated (see [1]–[9]). In [10], Reissing proved the existence of 2π-periodic solutions under the condition

$$N^2 + \varepsilon_0 \leq \frac{g(x)}{x} \leq (N+1)^2 - \varepsilon_0,$$

where $\varepsilon_0 > 0$, $N \in \mathbf{Z}_+$, and $|x|$ is sufficiently large.

Li and Wang[11] proved the existence of 2π-periodic solutions for higher order Duffing equations. Using the technique of Lazer[12] and Schauder's fixed point theorem, Cong *et al.*[13],[14] studied the existence and uniqueness of periodic solutions for the even and odd

*Received date: Oct. 12, 2007.
Foundation item: Partly by NSFC (10501017) under Grants.

order differential equation. By using degree theory, Liu and Li[15] investigated the existence of periodic solutions for more generalized higher order Duffing equations. Some other related results can be found in [1]–[9].

Analogously to the existence of periodic solutions, finding the periodic solutions is also a very important problem in many science fields (see [16]). The main aims of this paper is to present a result for the existence of periodic solutions for high order Duffing equations, and construct a global convergence algorithm to find the periodic solutions numerically.

In Section 2, we recall some basic facts on the homotopy method, and introduce two valuable lemmas. Section 3 is devoted to our main results. The construction of homotopy method and some examples are presented in Section 4.

2 Preliminary Results

We first present some preliminaries.

Lemma 2.1([17], Parametrized Sard Theorem) *Let $V \subset \mathbf{R}^k$, $U \subset \mathbf{R}^n$ be open sets, and $\Phi : V \times U \to \mathbf{R}^m$ a C^r map, where $r > \max\{0,\ n-m\}$. If $0 \in \mathbf{R}^k$ is a regular value of Φ, then for almost all $a \in V$, 0 is a regular value of $\Phi_a(\cdot) = \Phi(a, \cdot)$.*

Lemma 2.2[18] *Let $\Phi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ be a C^1 map and $0 \in \mathbf{R}^n$ a regular value of Φ. Then $\Phi^{-1}(0)$ is a C^1 manifold of dimension 1.*

Lemma 2.3[18] *A C^1 manifold of dimension 1 is C^1 homeomorphic to a loop or an interval (open, close, or semi-closed).*

Lemma 2.4[19] *Let $\Phi : \mathbf{R}^{n+1} \to \mathbf{R}^n$ be a C^1 homotopy and 0 a regular value of Φ. Then each solution $x(s)$ of the initial value problem*

$$\frac{\mathrm{d}x_i}{\mathrm{d}s} = (-1)^{i_0+i+1}\det \Phi_i', \qquad x_i(0) = x_{i_0},\ i = 1, \cdots, n+1$$

is a C^1 path in $\Phi^{-1}(0)$, where $i_0 \in \{0,\ 1\}$, s is parameter, and

$$\Phi_i' = (\Phi_{x_1}, \cdots, \Phi_{x_{i-1}}, \Phi_{x_{i+1}}, \cdots, \Phi_{x_{n+1}}), \qquad i = 1, \cdots, n+1.$$

Lemma 2.5[20] *Let X be a Banach space and $T \in \mathscr{L}(X)$. Then $\lim\limits_{n\to\infty} \|T^n\|^{1/n}$ exists and it is equal to $r(T)$. If X is a Hilbert space, and T is self-adjoint, then $r(T) = \|T\|$.*

Let

$$\lambda_0 = \frac{1}{2}[N^{2n} + (N+1)^{2n}], \qquad \mu_0 = \sqrt[2n]{\lambda_0}, \qquad \delta = \lambda_0 - N^{2n},$$

and

$$D \equiv \{x \in C^{2n-1}[0, 2\pi] :\ x^{(i)}(2\pi) = x^{(i)}(0),\ i = 0, 1, \cdots, 2n-1,$$
$$x^{(2n-1)}(t) \text{ is absolutely continuous on } [0, 2\pi],\ \text{and } x^{(2n)} \in L^2[0, 2\pi]\}.$$

Define $L : D \to L^2[0, 2\pi]$ by

$$Lx \equiv x^{(2n)} + (-1)^{n+1}\lambda_0 x.$$