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Linear Maps Preserving Idempotency of Products of Matrices on Upper Triangular Matrix Algebras*

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Abstract: Let $\mathcal{T}_n$ be the algebra of all $n \times n$ complex upper triangular matrices. We give the concrete forms of linear injective maps on $\mathcal{T}_n$ which preserve the nonzero idempotency of either products of two matrices or triple Jordan products of two matrices.

Key words: linear map, matrix, idempotent, product of two matrices, triple Jordan product of two matrices

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1 Introduction

The study of linear preserver maps on operator algebras preserving certain subsets, properties or relations has attracted many authors in the last few decades. Recently, there has been considerable interest in studying linear preserver problems concerning certain properties of products or triple Jordan products of operators (cf. [1]–[6]). Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$. In [3], the unital non-linear surjective maps on $\mathcal{B}(\mathcal{X})$ preserving nonzero idempotency of products of two operators in both directions are considered. On the other hand, linear maps preserving nonzero idempotency of products of two operators or triple Jordan products of operators in one direction are characterized in [6]. In this paper, we consider those linear maps preserving nonzero idempotency of products or triple Jordan products of two matrices on complex upper triangular matrix algebras. Let $\mathcal{M}_n$ be the algebra of all complex $n \times n$ matrices and let $\mathcal{T}_n$ (resp. $\mathcal{T}_n^0$) be the algebra of all complex $n \times n$ upper (resp. strictly upper) triangular matrices. We denote by I the identity in $\mathcal{M}_n$. Note that a matrix A in $\mathcal{T}_n$ is nilpotent if and only

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if $A \in \mathcal{T}_n^0$. We denote by $\mathbb{C}^n$ the n-dimensional complex Euclidian space. Let δ_{ij} be the Kronecker's numbers, that is,

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & \text{otherwise}, \end{cases} \qquad 1 \le i, j \le n.$$

Put

$$e_i = (\delta_{i1},\ \delta_{i2},\ \cdots,\ \delta_{in}), \qquad 1 \le i \le n.$$

Then $\{e_1,\ e_2,\ \cdots,\ e_n\}$ is the standard orthogonal basis of $\mathbb{C}^n$. We regard a matrix A as a linear transformation on $\mathbb{C}^n$. For every pair of vectors $x, y \in \mathbb{C}^n$, (x, y) denotes the inner product of x and y. The symbol $x \otimes y$ stands for the rank-1 matrix on $\mathcal{M}_n$ defined by

$$(x \otimes y)z = (z, y)x, \qquad z \in \mathbb{C}^n.$$

Then $\{E_{ij} = e_i \otimes e_j :\ i, j = 1, 2, \cdots, n\}$ is a basis of $\mathcal{M}_n$ while $\{E_{ij} = e_i \otimes e_j :\ 1 \le i \le j \le n\}$ is a basis of $\mathcal{T}_n$. For any complex numbers $b_1,\ b_2,\ \cdots,\ b_n$, we denote by $\mathrm{diag}(b_1,\ b_2,\ \cdots,\ b_n)$ the diagonal matrix. Let

$$J = \sum_{i=1}^{n} e_i \otimes e_{n-i+1}$$

be a canonical unitary matrix in $\mathcal{M}_n$. Rank-1 operator $x \otimes y$ is idempotent (resp. nilpotent) if $(x, y) = 1$ (resp. $(x, y) = 0$). For every pair of idempotents P and Q, we say $P \leqslant Q$ if $PQ = QP = P$ and we say $P < Q$ if $P \leqslant Q$ and $P \ne Q$. P and Q are orthogonal if $PQ = QP = 0$. Let

$$\mathcal{L} = \{\{e_1\},\ \{e_1,\ e_2\},\ \cdots,\ \{e_1,\ \cdots,\ e_n\}\}.$$

Then $\mathcal{L}$ is a canonical nest and

$$\mathcal{T}_n = \{A \in \mathcal{M}_n :\ AM \subseteq M,\ \forall M \in \mathcal{L}\}$$

is the nest algebra with nest $\mathcal{L}$.

In this paper, we consider linear injective maps φ on $\mathcal{T}_n$ which preserve nonzero idempotency of either products of two matrices or triple Jordan products of two matrices.

2 Linear Maps Preserving Nonzero Idempotency of Products of Two Matrices

Let φ be a linear map on $\mathcal{T}_n$ preserving nonzero idempotency of products of two matrices, that is, $\varphi(A)\varphi(B)$ is a nonzero idempotent matrix if AB is a nonzero idempotent matrix for all $A, B \in \mathcal{T}_n$ in this section. Our main result is the following theorem.

Theorem 2.1 *Let φ be a linear injective map on $\mathcal{T}_n$. Then φ preserves nonzero idempotency of products of two matrices if and only if there exist an invertible matrix $P \in \mathcal{T}_n$ and a constant λ with $\lambda^2 = 1$ such that one of the following forms holds*:

(1) *$\varphi(X) = \lambda P X P^{-1}$ for any $X \in \mathcal{T}_n$;*

(2) *$n = 2$ and $\varphi(X) = \lambda P J X^t J P^{-1}$ for any $X \in \mathcal{T}_n$, where X^t denotes the transpose of X.*