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Effective Maximum Principles for Spectral Methods

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Abstract. Many physical problems such as Allen-Cahn flows have natural maximum principles which yield strong point-wise control of the physical solutions in terms of the boundary data, the initial conditions and the operator coefficients. Sharp/strict maximum principles insomuch of fundamental importance for the continuous problem often do not persist under numerical discretization. A lot of past research concentrates on designing fine numerical schemes which preserves the sharp maximum principles especially for nonlinear problems. However these sharp principles not only sometimes introduce unwanted stringent conditions on the numerical schemes but also completely leaves many powerful frequency-based methods unattended and rarely analyzed directly in the sharp maximum norm topology. A prominent example is the spectral methods in the family of weighted residual methods.

In this work we introduce and develop a new framework of almost sharp maximum principles which allow the numerical solutions to deviate from the sharp bound by a controllable discretization error: we call them effective maximum principles. We showcase the analysis for the classical Fourier spectral methods including Fourier Galerkin and Fourier collocation in space with forward Euler in time or second order Strang splitting. The model equations include the Allen-Cahn equations with double well potential, the Burgers equation and the Navier-Stokes equations. We give a comprehensive proof of the effective maximum principles under very general parametric conditions.

AMS subject classifications: 35Q35

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1 Introduction

In solving physical problems such as Allen-Cahn flows in interfacial dynamics, the maximum principle plays an important role since it gives strong point-wise control of the physical solutions in terms of the boundary data, the initial conditions and the operator coefficients. For practical numerical simulations, it is often the case that sharp/strict maximum principles for the continuous problem often do not persist under numerical discretization. A lot of past research is centered on designing fine numerical schemes which preserves the maximum in a sharp way especially for nonlinear problems. For linear parabolic equations, it is well known that central finite difference in space with backward Euler time stepping can preserve the sharp maximum principle (cf. Chapter 9 of [5] for a textbook analysis of 1D homogeneous heat equation). This is also the case if one employs lumped mass linear finite element in space using acute simplicial triangulation. Although preserving the sharp maximum principle is highly desirable for numerical simulations, these often introduce unwanted stringent conditions on the numerical schemes. Moreover it completely leaves out many powerful L^2-based methods unattended and rarely analyzed directly in the sharp maximum norm topology. In this respect a prominent example is the spectral methods in the family of weighted residual methods. In this work we introduce and develop a new framework of almost sharp maximum principles which allow the numerical solutions to deviate from the sharp bound by a controllable discretization error: we call them effective maximum principles. Our main models are Allen-Cahn equations in physical dimensions $d\leq 3$, but we also discuss related models such as Burgers equations, Navier-Stokes equations. All these will be discussed in this introduction.

We begin by considering the Allen-Cahn equation in physical dimensions $d=1,2,3$:

$$\begin{cases} \partial_t u=\nu^2\Delta u-f(u), & t>0, \\ u\big|_{t=0}=u_0, \\ \text{periodic boundary conditions}, \end{cases} \tag{1.1}$$

where u is a scalar function which typically represents the concentration of one of the two metallic components of the alloy. For simplicity we consider the periodic boundary condition and assume the function to have period 1 in each spatial coordinate axis. The parameter $\nu>0$ controls the interfacial width which is small compared with the system size under study. The nonlinear term has the usual double well