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A Sequential Least Squares Method for Elliptic Equations in Non-Divergence Form

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Abstract. We develop a new least squares method for solving the second-order elliptic equations in non-divergence form. Two least-squares-type functionals are proposed for solving the equation in two sequential steps. We first obtain a numerical approximation to the gradient in a piecewise irrotational polynomial space. Then together with the numerical gradient, we seek a numerical solution of the primitive variable in the continuous Lagrange finite element space. The variational setting naturally provides an a posteriori error which can be used in an adaptive refinement algorithm. The error estimates under the L^2 norm and the energy norm for both two unknowns are derived. By a series of numerical experiments, we verify the convergence rates and show the efficiency of the adaptive algorithm.

AMS subject classifications: 65N30

Key words: Non-divergence form, least squares method, piecewise irrotational space, discontinuous Galerkin method.

1. Introduction

This work is concerned with the non-divergence form second-order elliptic equation, which is often encountered in many applications from areas such as probability and stochastic processes [20]. In addition, such problems also naturally arise as the linearization to fully nonlinear PDEs, as obtained by applying the Newton's iterative method, see [7, 9]. Due to the non-divergence structure, it is invalid to derive a variational formulation by applying the integration by parts. Instead, the existence and uniqueness of the solutions to this problem are sought in the strong sense, we refer to [2, 8, 11, 20, 21] and the references therein for the well-posedness of the solutions to the non-divergence form second-order elliptic equation.

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Recently several finite element methods have been proposed, though such a problem does not naturally fit within the standard Galerkin framework. Conforming finite element methods require H^2-regularity for approximating the strong solution, which naturally leads to a C^1 finite element space [4, 6]. But the C^1 finite elements are sometimes considered impractical. In [15], the authors introduced a mixed finite element method with C^0 finite element space via a finite element Hessian obtained in the same approximation space. In [8], the authors proposed and analyzed a finite element method with C^0 space by introducing an interior penalty term. But the coefficient matrix is assumed to be continuous. Gallistl introduced a conforming mixed finite element method based on a least squares functional, we refer to [11] for more details. In [18], the authors proposed a simple and convergent finite element method with C^0 finite element space. Based on discontinuous approximations, Smears and Süli proposed a discontinuous Galerkin method where the optimal convergence rate in h with respect to broken H^2 norm is proven and the authors have extended this method to the Hamilton-Jacobi-Bellman equations [12,21]. In addition, we refer to [10,19,22] for more methods to this problem.

In this paper, we propose a new least squares finite element method for solving the non-divergence elliptic problem. We rewrite the equation into an equivalent first-order system as a fundamental requirement in the least squares method [5]. We employ two different approximation spaces to solve the gradient and the primitive variable sequentially, which is motivated from the idea in [16]. We first define a least squares functional to seek a numerical approximation to the gradient in a piecewise irrotational polynomial space. Then we obtain the approximation to the primitive variable with the numerical gradient by solving another least squares problem in the standard C^0 finite element space. Our method avoids solving a saddle-point problem of mixed formulation, and in contrast to [12, 18, 20] our method only involves the first-order operator in each step. We prove the convergence rates for both variables in L^2 norm and energy norm. The least squares functional naturally serves as an a posteriori error estimate and we introduce an adaptive algorithm for solving the problem of low regularity. By carrying out a series of numerical experiments, we verify the convergence orders in the error estimates and illustrate the efficiency of the adaptive algorithm.

The rest of this paper is organized as follows. Section 2 gives the notations that will be used throughout the paper and defines the considered problem. In Section 3, we introduce the piecewise irrotational approximation space and give some basic properties of this space. In Section 4, we propose the least squares method for both two variables respectively and the error estimates are derived. In Section 5, a series of numerical experiments are presented for testing the accuracy of the proposed scheme.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a convex polygonal (polyhedral) domain with the boundary $\partial\Omega$. We denote by $\mathcal{T}_h$ a regular and shape-regular subdivision of Ω into simplexes. Let $\mathcal{E}_h^i$ be the set of all interior faces associated with the subdivision $\mathcal{T}_h$, $\mathcal{E}_h^b$ the set of all