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Boundedness of Area Functions Related to Schrödinger Operators and Their Commutators in Weighted Hardy Spaces

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. In this paper, we consider the area function S_Q related to the Schrödinger operator $\mathcal{L}$ and its commutator $S_{Q,b}$, establish the boundedness of S_Q from $H^p_\rho(w)$ to $L^p(w)$ or $WL^p(w)$, as well as the boundedness of $S_{Q,b}$ from $H^1_\rho(w)$ to $WL^1(w)$.

Key Words: Area functions, Schrödinger operator, weighted Hardy space.

AMS Subject Classifications: 42B25, 42B20

1 Introduction

Throughout this paper, $\mathcal{L}$ always denotes the following Schrödinger differential operator

$$\mathcal{L} = -\Delta + V(x) \quad \text{on } \mathbb{R}^n, \quad n \geq 3,$$

where V is a nonnegative potential belongs to reverse Hölder class $RH_{n/2}$ (see Section 1.2).

The study of the Schrödinger operator $\mathcal{L}$ has recently attracted much attention, see [1, 2, 4, 5, 12, 19]. In particular, Shen [12] proved the Schrödinger type operators, such as $\nabla(-\Delta+V)^{-1}\nabla$, $\nabla(-\Delta+V)^{-1/2}$, $(-\Delta+V)^{-1/2}\nabla$ with $V \in RH_n$, and $(-\Delta+V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in RH_{n/2}$, are standard Calderón-Zygmund operators.

In 2011, Bongioanni, etc. [1] introduced a new space of functions $BMO_\theta(\rho)$ as a generalization of the classical BMO space. They [2] also introduced a new weight class $A_q^{\rho,\theta}$ that

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locally behaves as Muckenhoupt's one and actually contains that. Both $BMO_\theta(\rho)$ and $A_q^{\rho,\theta}$ are associated with the potential V. The authors [1,2] also established $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ boundedness for commutators of Riesz transforms associated with Schrödinger operators with $BMO_\theta(\rho)$ functions, and weighted boundedness for Riesz transforms, fractional integrals and Littlewood-Paley functions related to Schrödinger operator with $A_q^{\rho,\theta}$ weights. Recently, Tang, etc. [14–16] established weighted norm inequalities for some Schrödinger type operators, including commutators of Riesz transforms, fractional integrals, Littlewood-Paley functions and area functions related to Schrödinger operators, etc.

On the other hand, the function spaces related to $\mathcal{L}$ has attracted wide concern for years. In 1999, Dziubański and Zienkiewicz [4] defined the Hardy space related to $\mathcal{L}$, and gave some equivalent characterizations. In 2005, Dziubański, etc [5] defined the BMO space related to $\mathcal{L}$, and proved that it is the dual space of the above Hardy space. The weighted version of these theory have been also considered recently; see [8,13]. It should be pointed out that in [8,13], the authors considered the weight functions belonging to Munckhoupt weight class. Very recently, Tang and Zhu [17] studied the properties of weighted Hardy spaces with $A_q^{\rho,\theta}$ weights.

In this paper, we continue to study weighted norm inequalities for area functions related to Schrödinger operators and their commutators. In fact, the weights we consider here are $A_q^{\rho,\theta}$ weights, and the weighted boundedness are of the type $H_\rho^p(\omega) \to L^p(\omega)$ or $WL^p(\omega)$, where $H_\rho^p(\omega)$ denotes weighted Hardy space related to ρ.

We first introduce some definitions. The area S_Q function related to $\mathcal{L}$ is defined by

$$S_Q(f)(x) := \left(\int_0^\infty \int_{|x-y|<t} |Q_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \tag{1.1}$$

where

$$(Q_t f)(x) := t^2 \left(\frac{dT_s}{ds}\bigg|_{s=t^2} f \right)(x), \quad T_s = e^{-sL}, \quad (x,t) \in \mathbb{R}_+^{n+1} = (0,\infty) \times \mathbb{R}^n. \tag{1.2}$$

The commutator of S_Q with $b \in BMO_\theta(\rho)$ is defined by

$$S_{Q,b}(f)(x) := \left(\int_0^\infty \int_{|x-y|<t} |Q_t((b(x) - b(\cdot))f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \tag{1.3}$$

The main results of this paper are as follows.

Theorem 1.1. *Let $q \geq 1$ and $w \in A_q^{\rho,\infty}$,*

(i) if $\frac{n}{n+\delta_0} < p \leq 1$ and $1 \leq q \leq p\left(1 + \frac{\delta_0}{n}\right)$, then for all $f \in H_\rho^p(\omega)$,

$$\|S_Q f\|_{L^p(w)} \lesssim \|f\|_{H_\rho^p(\omega)};$$