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A New Relaxed Splitting Preconditioner for Multidimensional Multi-Group Radiation Diffusion Equations

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Abstract. Motivated by the ideas of Frigo *et al.* [SIAM J. Sci. Comput. 41 (2019) B694–B720], we develop a novel relaxed splitting preconditioner and consider its parallel implementation. Fully-coupled fully-implicit linearised algebraic systems arising from the multidimensional multi-group radiation diffusion equations are solved by using algebraic multigrid subsolvers. Spectral properties of the relaxed splitting right-preconditioned matrix are studied. This allows to introduce an easily implementable algebraic selection strategy for finding the corresponding relaxation parameter. Numerical experiments show that the new preconditioner outperforms some existing popular preconditioners in robustness and efficiency and is well scalable both algorithmically and in parallel.

AMS subject classifications: 65F10, 65N55, 65Y05, 65Z05

Key words: Radiation diffusion equations, relaxed splitting, algebraic multigrid, incomplete LU factorization, parallel computing.

1. Introduction

The radiation transport processes describe the transmission, scattering and interaction of photons in vacuum or multicomponent background mediums. They are used in various coupled multiphysics applications such as astrophysical phenomena, biomedicine, and

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laser indirect-drive inertial confinement fusions (ICFs). The simulations of the radiation transport are very difficult because of a highly nonlinear behavior in optically thick mediums. In such circumstances, the radiation transport is modeled by a flux-limited time-dependent highly nonlinear and discontinuous multi-group radiation diffusion (MGD) approximation [4, 23]. The corresponding analysis is based on the efficient solution of fully-coupled sparse systems of linear algebraic equations with the degree of freedom (DoF) ranging from 10^7 to 10^{11}. Implicit time integration approaches are desired to overcome time-step constraints. In order to evaluate the nonlinear parts at the previous nonlinear iteration level, Kačanov [29] employed the method of frozen coefficients. An important observation is that because of the complex volatile nonlinear coupling of various physical quantities from numerous interacting spatial and temporal scales, the MGD equations are often discretised by finite volume schemes [14,24,38,45,47,52]. This leads to the series of nonsymmetric but positive definite linear systems, which have to be solved at each time-step and/or nonlinear iterations. We note that the finding of the corresponding numerical solutions is time-consuming, mainly because the coefficient matrices are highly ill-conditioned and their conditioning deteriorates as the mesh-size gets smaller. It takes more than 80% of the entire ICF simulation time in general. We note that the memory requirements of direct linear solvers severely limit the number of frequency groups and spatial mesh-size that can be tackled [3,16,30]. In order to deal with this challenging task and to exploit the ever-increasing computing power, robust and scalable iterative linear solvers must be combined with the preconditioners requiring minimal user input. The state-of-the-art iterative methods are the Krylov subspace solvers — cf. [26, 34, 42, 48]. The convergence of such methods is usually based on the conditioning of an associated matrix and on the clustering of its eigenvalues.

Over the past two decades, a variety of block preconditioners (also called physics-based preconditioners) have been developed in coupled multiphysics PDE applications, where system matrices have an underlying block structure. Block preconditioners are usually used in order to split a problem into a series of subproblems (which can be easily solved) and to establish an object-oriented framework incorporating off-the-shelf single-physics solvers and preconditioners. Thus substantial efforts have been spent on physics-based preconditioning strategies with algebraic multigrid (AMG) as an essential ingredient. This approach turned out to be successful in various coupled multiphysics PDE simulations — e.g. in poroelasticity [1], geophysical electromagnetics [8], Cahn-Hilliard Navier-Stokes systems [9], multiphase poromechanics of heterogeneous media [11], linear elasticity in mixed form [13], incompressible (reduced) resistive magnetohydrodynamic [15], incompressible flow simulations [21], elliptic optimal control problems [22], fluid-structure interaction problems in hemodynamics [32], models of coupled magma/mantle dynamics [39] and incompressible Navier-Stokes problems [44]. For fully implicit discretisations of multidimensional radiation diffusion equations, An *et al.* [2], Feng *et al.* [18], and Mousseau *et al.* [33] proposed various operator-based preconditioners in the Jacobian-free Newton-Krylov framework. Brown and Woodward [10] considered the Schur complement preconditioned generalised minimum residual (GMRES) solver without restarting based on higher-order time integration. Shu *et al.* [46] constructed theoretical and practical lower and upper block tri-