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Lie Symmetry Analysis and Wave Propagation in Variable-Coefficient Nonlinear Physical Phenomena

Mohamed R. Ali[1], Wen-Xiu Ma[2,3,4,5,*] and R. Sadat[6]

[1] *Department of Mathematics, Faculty of Engineering, Benha University, Egypt.*
[2] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China.*
[3] *Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia.*
[4] *Department of Mathematics and Statistics, University of South Florida, Tampa, FL33620-5700, USA.*
[5] *School of Mathematics, South China University of Technology, Guangzhou 510640, China.*
[6] *Department of Mathematics, Zagazig Faculty of Engineering, Zagazig University, Zagazig, Egypt.*

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Abstract. We present Lie symmetry analysis to explore solitary wave solutions, two-soliton type solutions and three-soliton type solutions in variable-coefficient nonlinear physical phenomena. An example is a (2+1)-dimensional variable-coefficient Bogoyavlensky-Konopelchenko (VCBK) equation. We compute the Lie algebra of infinitesimals of its symmetry vector fields and an optimal system of one-dimensional sub-Lie algebras of the resulting symmetries. Two stages of Lie symmetry reductions will be built to reduce the VCBK equation to nonlinear ordinary differential equations (ODEs) and new analytical solutions to those ODEs will be found by using the integration method. Some of such resulting solutions to the VCBK equation and their dynamics will be illustrated through three-dimensional plots.

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Key words: Symmetry analysis, partial differential equations, the variable coefficients (2+1)-dimensional Bogoyavlensky-Konopelchenko equation.

1. Introduction

The (2+1) dimensional Bogoyavlensky-Konopelchenko (BK) equation reads

$$w_t+\alpha w_{xxx}+\beta w_{xxy}+6\alpha w w_x+4\beta w w_y+4\beta w_x\partial_x^{-1}w_y=0, \tag{1.1}$$

*Corresponding author. *Email address:* `mawx@cas.usf.edu` (W.X. Ma)

where α and β are arbitrary constants and ∂_x^{-1} is the integral with respect to x, cf. [1,2,16]. For simplicity, substituting $w(x,y,t)=v_x$, we get

$$v_{xt}+\alpha v_{xxxx}+\beta v_{xxxy}+6\alpha v_x v_{xx}+4\beta v_x v_{xy}+4\beta v_y v_{xx}=0. \tag{1.2}$$

Many kinds of research [7, 22] depicted that $v(x,y,t)$ describes the interaction of a Riemann wave propagating along y-axis and x-axis. From our fast review, there are many presented solutions for (1.2). Ray [15, 16] applied the Lie symmetry analysis to present some generators through the prolongation theorem and the geometric approach, respectively, to reduce (1.2) to ODEs and generate exact solutions. The authors in [9] presented some modifications to Ray's works. Besides, the authors in [23] applied the Lie symmetry method to (1.2) and investigated its conservation laws. Some lump solutions and interacted soliton solutions had been obtained using the Hirota bilinear method in [5, 6, 8, 11, 12, 14, 18]. On the other hand, some authors studied the variable-coefficient Bogoyavlensky-Konopelchenko (VCBK) equation

$$v_{xt}+\alpha(t)v_{xxxx}+\beta(t)v_{xxxy}+\gamma(t)v_x v_{xx}+\delta(t)v_x v_{xy}+\theta(t)v_y v_{xx}=0, \tag{1.3}$$

where $\alpha(t)$, $\beta(t),\gamma(t),\delta(t)$ and $\theta(t)$ are real functions in time and $\gamma(t)=6\alpha(t)$ and $\delta(t)=\theta(t)=4\beta(t)$ [5,9,13,23,24].

The authors of [1] stratified the Hirota method and Bell polynomials to demonstrate the solitons solutions for Eq. (1.3). The interaction between solitons was demonstrated in [13] through utilizing the unified method. Numerous research efforts explore diverse solutions of Eq. (1.3) by various methods such as the inverse scattering method with the aid of Lax pairs and the G'/G expansion technique [20–22].

In this paper, we study the VCBK equation, based on the associated Lie algebra [3, 5, 10, 19–23]. Eq. (1.3) has four obscure vectors which were demonstrated in [1], and so we perfect the Lie vectors consisting of arbitrary functions of time through a commutative product for different magnitudes of functions and set the same steps for constant coefficients in Eq. (1.2). Over one or two steps of reduction, several ODEs that have no square are disbanded using the integrating factors as in [17].

In Section 2, we demonstrate three cases for Eq. (1.3). In each case, we use a different value for $\alpha(t),\beta(t),\gamma(t),\delta(t)$ and $\theta(t)$. We examine new solitons and other solutions for the VCBK equation in Section 3 using new Lie vectors that we got in the previous section. In Section 4, we finalise the action by conclusion.

2. Infinitesimal Generators of Lie Symmetries

We present three cases, in each one we use a different value of the real function in (1.3). Through the commutative product between the resulted vectors we investigate in each case new infinitesimals for the VCBK equation.