Journal of Computational Mathematics
Vol.40, No.1, 2022, 1–25.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2005-m2019-0304

THEORETICAL ANALYSES ON DISCRETE FORMULAE OF DIRECTIONAL DIFFERENTIALS IN THE FINITE POINT METHOD*

Guixia Lv[1)] and Longjun Shen
Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P. O. Box 8009-26, Beijing 100088, China
Email: lv_guixia@iapcm.ac.cn, shenlj@iapcm.ac.cn

Abstract

For the five-point discrete formulae of directional derivatives in the finite point method, overcoming the challenge resulted from scattered point sets and making full use of the explicit expressions and accuracy of the formulae, this paper obtains a number of theoretical results: (1) a concise expression with definite meaning of the complicated directional difference coefficient matrix is presented, which characterizes the correlation between coefficients and the connection between coefficients and scattered geometric characteristics; (2) various expressions of the discriminant function for the solvability of numerical differentials along with the estimation of its lower bound are given, which are the bases for selecting neighboring points and making analysis; (3) the estimations of combinatorial elements and of each element in the directional difference coefficient matrix are put out, which exclude the existence of singularity. Finally, the theoretical analysis results are verified by numerical calculations.

The results of this paper have strong regularity, which lay the foundation for further research on the finite point method for solving partial differential equations.

Mathematics subject classification: 65D25, 65N06, 65N35.
Key words: Finite point method, Finite difference, Scattered point distribution, Discrete directional differentials, Theoretical analysis.

1. Introduction

The finite difference method for solving partial differential equations (PDEs) (see, e.g., [1–4]) was originated in the 1920s, which was constructed on regular grids in computational domains. Due to the limitation of computational problems and conditions at that time, the computational scale was often small, and the complexity was not high, so the method could solve problems effectively. However, with the emergence of large and complex problems, the traditional finite difference method on regular grids was facing enormous challenges. To settle the matter, [1] and [5] proposed the method of dividing irregular mesh regions into regular sub-regions, respectively. [6] considered the finite difference method on irregular sub-regions with restricted topology. For irregular grids, [7] proposed a finite difference method with six-point stencils earlier, which could give approximations up to second-order derivatives, but was often troubled by singularity or ill-conditioning of numerical differentials.

* Received December 21, 2019 / Revised version received April 18, 2020 / Accepted May 12, 2020 / Published online June 18, 2021 /
1) Corresponding author

Thereafter, to overcome the singularity, some scholars proposed the generalized finite difference method by enlarging stencils [8–10], which enhanced the computational capability of the finite difference method on irregular grids to a certain extent [11, 12]. With insight into the finite difference method, it is not difficult to find from the discrete process that it has properties of meshless methods.

In recent years, many efforts have been devoted to generalize the traditional difference method on scattered point sets. Most of them are based on a large number of neighboring points to fit derivatives [13–16], while there are also a few jobs in which a small number of neighbors are employed [17–19], however, none of them can address the issue of singularity or ill-conditioning of numerical derivatives at the fundamental level. In addition, many scholars have studied the difference method based on radial basis function (RBF) [20–24], which can be also viewed as a generalization of the traditional difference method.

To sum up, these works on scattered point sets undoubtedly improve the computational ability of the difference method, whereas scattered point sets also bring great difficulties to the theoretical analysis of related methods. Compared with its wide applications, theoretical results of meshless finite difference method are far from enough, especially for discrete analyses of PDEs. At present, the few existing works are often limited on discrete points with special distributions or on them only with a small number of discrete points near the boundary being irregular [25, 26].

The finite point method [27] to be studied in this paper is the finite difference method based on scattered point sets in irregular regions. In this method, only a few neighboring points are required to give the discretization of differential operators with higher accuracy. For example, in the two-dimensional case, given a discrete point, only 5 neighbors are demanded to obtain second-order approximations for first-order derivatives and first-order approximations for second-order derivatives. Above all, based on the analysis of the solvability conditions of numerical derivatives, the method of selecting neighboring points is presented, and the software module is formed, which can overcome the singularity problem of numerical derivatives fundamentally. The explicit expression of derivative approximation is also derived, by which some important theoretical results, such as convergence analysis for Poisson equation on scattered point sets [28] and the compatibility of nonlinear diffusion operators [29] are obtained, moreover, by which two-dimensional three-temperature energy equations in high temperature plasma physics have also been successfully solved numerically [29]. This paper aims at making a detailed analysis to give estimations of the numerical discrete formulae of directional derivatives on scattered point sets, including the lower bound of absolute value of the discriminant function for the solvability of numerical derivatives and the bounds of a series of coefficients. These theoretical results are the key bases for further developing the theoretical research of relevant methods.

The rest of the paper is arranged as follows: Section 2 gives some basic denotations; Section 3 discusses the structure of the directional difference coefficient matrix; Section 4 presents a variety of expressions and estimations of the discriminant function for the solvability of numerical differentials; Section 5 puts out analyses and estimation results of the directional difference coefficient matrix; Section 6 validates the theoretical results by numerical examples. Finally, the conclusions are drawn in Section 7.