Journal of Computational Mathematics
Vol.40, No.1, 2022, 70–88.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2008-m2018-0218

LONG-TIME OSCILLATORY ENERGY CONSERVATION OF TOTAL ENERGY-PRESERVING METHODS FOR HIGHLY OSCILLATORY HAMILTONIAN SYSTEMS*

Bin Wang[1)]
School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China
Email: wangbinmaths@xjtu.edu.cn
Xinyuan Wu
Department of Mathematics, Nanjing University, Nanjing 210093, China
School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China
Email: xywu@nju.edu.cn

Abstract

For an integrator when applied to a highly oscillatory system, the near conservation of the oscillatory energy over long times is an important aspect. In this paper, we study the long-time near conservation of oscillatory energy for the adapted average vector field (AAVF) method when applied to highly oscillatory Hamiltonian systems. This AAVF method is an extension of the average vector field method and preserves the total energy of highly oscillatory Hamiltonian systems exactly. This paper is devoted to analysing another important property of AAVF method, i.e., the near conservation of its oscillatory energy in a long term. The long-time oscillatory energy conservation is obtained via constructing a modulated Fourier expansion of the AAVF method and deriving an almost invariant of the expansion. A similar result of the method in the multi-frequency case is also presented in this paper.

Mathematics subject classification: 65P10, 65L05.
Key words: Highly oscillatory Hamiltonian systems, Modulated Fourier expansion, AAVF method, Energy-preserving methods, Long-time oscillatory, Energy conservation.

1. Introduction

This paper is concerned with the long-time oscillatory energy behaviour of energy-preserving methods for the highly oscillatory Hamiltonian system

$$\begin{cases} \dot{q} = \nabla_p H(q,p), & q(0) = q_0, \\ \dot{p} = -\nabla_q H(q,p), & p(0) = p_0, \end{cases} \tag{1.1}$$

where the Hamiltonian function is given by

$$H(q,p) = \frac{1}{2}\left(\|p\|^2 + \|\Omega q\|^2 \right) + U(q). \tag{1.2}$$

Here $U(q)$ is a real-valued function. According to the partition of the square matrix

$$\Omega = \begin{pmatrix} 0_{d_1\times d_1} & 0_{d_2\times d_2} \\ 0_{d_1\times d_1} & \omega I_{d_2\times d_2} \end{pmatrix} \tag{1.3}$$

* Received October 4, 2018 / Revised version received June 22, 2020 / Accepted August 4, 2020 / Published online September 15, 2021 /
1) Corresponding author

with a large positive parameter ω, the vectors $p = (p_1, p_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $q = (q_1, q_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ are partitioned accordingly. As is known, the oscillatory energy of the system (1.1) is

$$I(q,p) = \frac{1}{2}p_2^\intercal p_2 + \frac{1}{2}\omega^2 q_2^\intercal q_2 \tag{1.4}$$

and it is nearly conserved over long times along the solution of (1.1) (see [17]). Our attention of this paper will particularly focus on the near conservation of the oscillatory energy (1.4) for energy-preserving methods over long-time intervals.

In order to preserve the total energy of Hamiltonian systems exactly by numerical methods, energy-preserving (EP) methods have been proposed and researched. In the recent decades, various kinds of EP methods have been derived, such as the average vector field (AVF) method [4, 5, 31], discrete gradient methods [26, 27], the energy-preserving collocation methods [7, 13], Hamiltonian Boundary Value Methods (HBVMs) [2, 3], energy-preserving exponentially-fitted methods [29,30], and time finite elements methods [1,22,37]. By taking advantage the frequency matrix of second-order highly oscillatory systems, a novel adapted AVF (AAVF) method has been formulated and studied in [36, 41] for the highly oscillatory Hamiltonian system (1.2). It has been proved in [36, 41] that this AAVF method exactly preserves the total energy (1.2) and it reduces to the AVF method when the frequency matrix vanishes. However, most existing publications dealing with EP methods focus on the formulation of the methods and the analysis of the EP property. It seems that the long-time behaviour of AAVF method concerning other structure-preserving aspects has never been studied in the literature, such as the long-time numerical conservation of oscillatory energies. As is well known that an important property of highly oscillatory systems is the near conservation of the oscillatory energy over long times.

On the other hand, in the recent two decades, modulated Fourier expansion has been presented and developed as an important mathematical tool in the study of the long-time behaviour for numerical methods/differential equations (see, e.g. [6, 9, 11, 16, 39]). It was firstly given in [15] and has been used in the long-time analysis for various numerical methods, such as for the Störmer–Verlet method in [14, 16], for trigonometric integrators in [6, 17], for an implicit-explicit method in [28, 33], for heterogeneous multiscale methods in [32], for splitting methods in [10, 12] and for a filtered Boris method in [18]. However, it is noted that, until now, the technique of modulated Fourier expansions has not been well applied to the long-term analysis for energy-preserving method in the literature. Very recently, the authors of [35] studied long-time momentum and actions behaviour of energy-preserving methods for semilinear wave equations.

Based on the facts stated above, the main contribution of this paper is to analyse the long-time oscillatory energy conservation for the AAVF method. To this end, the technique of modulated Fourier expansions with some adaptations will be used in the analysis. To our knowledge, this paper is the first one that rigorously studies the remarkable long-time oscillatory energy conservation of EP methods on highly oscillatory Hamiltonian systems by using modulated Fourier expansions.

The rest of this paper is organised as follows. We first present the scheme of AAVF method and carry out an illustrative numerical experiment in Section 2. Section 3 derives the modulated Fourier expansion of the AAVF method and analyse the bounds of the modulated Fourier functions. In Section 4, we show an almost invariant of the modulation system and then the main result concerning the long-time oscillatory energy conservation of AAVF method is derived. Section 5 extends the analysis to multi-frequency case and studies the long-time conservation of AAVF method when applied to multi-frequency highly oscillatory Hamiltonian systems. The last section includes the concluding remarks of this paper.