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ADAPTIVE AND OPTIMAL POINT-WISE ESTIMATIONS FOR DENSITIES IN GARCH-TYPE MODEL BY WAVELETS*

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Abstract

This paper considers adaptive point-wise estimations of density functions in GARCH-type model under the local Hölder condition by wavelet methods. A point-wise lower bound estimation of that model is first investigated; then we provide a linear wavelet estimate to obtain the optimal convergence rate, which means that the convergence rate coincides with the lower bound. The non-linear wavelet estimator is introduced for adaptivity, although it is nearly-optimal. However, the non-linear wavelet one depends on an upper bound of the smoothness index of unknown functions, we finally discuss a data driven version without any assumptions on the estimated functions.

Mathematics subject classification: 42C40, 62G07, 62G20.
Key words: Wavelets, Point-wise risk, Thresholding, Data-driven, GARCH-type model.

1. Introduction

The density estimation for the GARCH-type (Generalized Autoregressive Conditionally Heteroskedastic) model plays an important role in both statistics and econometrics [5]. In this current paper, we consider that density estimation model which can be described by the following mathematical model:

$$Y = XZ, \tag{1.1}$$

where X and Z are independent random variables. More precisely, the unknown density function f of X is to be estimated and *supp* $f \subseteq [0,1]$, while the density of Z is known. In general, we suppose that

$$Z := \prod_{i=1}^{v} U_i,$$

where v is a positive integer and $U_1, \cdots, U_v$ are independent and identically distributed (i.i.d.) random variables with uniform distribution $U_{(0,1)}$. When $v = 1$, model (1.1) reduces to the standard multiplicative concerning model, for which is sometimes called generalized multiplicative censoring model [1–3, 18, 19]. The purpose is to find an estimator $\widehat{f}_n$ based on the i.i.d. observed data $Y_1, \cdots, Y_n$ of Y approximating the unknown density f in some sense.

For the GARCH-type model, lots of literatures have been done the density estimations over L^p-risk by wavelet methods [4, 6–9, 16]. Asymptotic properties of the kernel estimators for a density derivative have been considered earlier in [15], while the performance of wavelet estimator was discussed in [16]. In 2012, Chesneau & Doosti [9] investigated the wavelet estimation of

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a density in GARCH model under various dependence structures. One year later, Chesneau [8] provided the upper bounds over L^2-risk of wavelet density estimation for GARCH-type model. Rao [16] considered L^2-risk estimation for the derivative of a density in GARCH-type model over Besov balls by wavelets in 2017. After two years, Cao & Wei [4] extended Rao's result to L^p-risk ($1 \le p < \infty$).

In contrast to the above L^p-risk estimation, we consider point-wise risk estimations for model (1.1) in this paper, because it is more concerned in some applications. For a density function set Σ, the maximal point-wise risk at $x \in \mathbb{R}$ over Σ means that

$$R_{p,n}(\widehat{f}_n, \Sigma, x) := \sup_{f\in\Sigma}\left[E\left|\widehat{f}_n(x) - f(x)\right|^p\right]^{\frac{1}{p}}$$

with $1 \le p < \infty$ and EX being the expectation of X. An estimator $\widehat{f}_n^*$ is said to be the optimal over Σ, if

$$R_{p,n}(\widehat{f}_n^*, \Sigma, x) \lesssim \inf_{\widehat{f}_n} R_{p,n}(\widehat{f}_n, \Sigma, x),$$

where the infimum runs over all possible estimator of $f \in \Sigma$. Here and throughout, $A \lesssim B$ denotes $A \le cB$ for some independent constant $c > 0$; $A \gtrsim B$ means $B \lesssim A$; $A \sim B$ stands for both $A \lesssim B$ and $A \gtrsim B$. Clearly,

$$R_{p,n}(\widehat{f}_n^*, \Sigma, x) \ge \inf_{\widehat{f}_n} R_{p,n}(\widehat{f}_n, \Sigma, x)$$

holds automatically.

It is more reasonable to estimate $f(x_0)$ (for fixed $x_0 \in \mathbb{R}$) under some smoothness of f in a neighborhood Ω_{x_0} of x_0 instead of $\mathbb{R}$. For a function f on $\mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$, we introduce the local Hölder condition of order s ($0 < s \le 1$) at x_0 in the sense that with $C > 0$ and Ω_{x_0},

$$|f(y) - f(x)| \le C|y - x|^s \tag{1.2}$$

holds for each $x, y \in \Omega_{x_0}$. We use $H^s(\Omega_{x_0})$ to denote all those functions satisfying (1.2) with a fixed constant $C > 0$.

For $s = N + \delta$ with $\delta \in (0, 1]$ and $N \in \mathbb{N}$, define

$$H^s(\Omega_{x_0}) := \left\{f,\ f^{(N)} \in H^\delta(\Omega_{x_0})\right\}.$$

More properties and advantages of the local Hölder space $H^s(\Omega_{x_0})$ can be found in Ref. [14,20]. Moreover, $H^s(\Omega_{x_0},\ M)$ stands for

$$H^s(\Omega_{x_0},\ M) := \{f \in H^s(\Omega_{x_0}),\ f \text{ is a density and } \|f\|_\infty \le M\}.$$

Obviously, $H^s(\Omega_{x_0},\ M) \subset L^2(\mathbb{R})$, because

$$\int_{\mathbb{R}} |f(x)|^2 dx \le \|f\|_\infty \|f\|_1 \le M\|f\|_1 < \infty.$$

This paper is organized as follows. In Section 2, we shall give a lower bound estimation of a density in GARCH-type model. Let $\widehat{f}_n$ be an estimator of a function in $H^s(\Omega_{x_0},\ M)$. Then with $1 \le p < \infty$,

$$\sup_{x\in\Omega_{x_0}} \inf_{\widehat{f}_n} \sup_{f\in H^s(\Omega_{x_0},\ M)} \left[E\left|\widehat{f}_n(x) - f(x)\right|^p\right]^{\frac{1}{p}} \gtrsim n^{-\frac{s}{2s+2v+1}}.$$