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On Smooth Solutions to the Thermostated Boltzmann Equation with Deformation

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Abstract. This paper concerns a kinetic model of the thermostated Boltzmann equation with a linear deformation force described by a constant matrix. The collision kernel under consideration includes both the Maxwell molecule and general hard potentials with angular cutoff. We construct the smooth steady solutions via a perturbation approach when the deformation strength is sufficiently small. The steady solution is a spatially homogeneous non Maxwellian state and may have the polynomial tail at large velocities. Moreover, we also establish the long time asymptotics toward steady states for the Cauchy problem on the corresponding spatially inhomogeneous equation in torus, which in turn gives the non-negativity of steady solutions.

AMS subject classifications: 35Q20, 35B40

Key words: Boltzmann equation, deformation force, thermostated force, non-equilibrium steady state, asymptotic stability.

1 Introduction

The homoenergetic solutions to the Boltzmann equation were first introduced by Galkin [19] and Truesdell [28] independently at almost the same time. These

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prototypical solutions not only indicate the existence of invariant manifolds of molecular dynamics but also give a new insight into the relation between atomic forces and nonequilibrium behavior of the gas. Recently, James *et al.* [25–27] and Bobylev *et al.* [10] provided the systematic mathematical study of the subject. Motivated by those works, the authors of this paper also considered the smoothness and asymptotic stability of self-similar solutions to the Boltzmann equation for the uniform shear flow in case of the Maxwell molecule [14]. In the non Maxwell molecule case, for instance, for the hard potentials, the problem is more subtle to treat and still remains largely open, because the temperature of system increases only in a polynomial rate depending on the collision kernel and the shear rate in the rescaled equation is no longer a constant but a time-dependent function, see the conjecture in [25] for details.

On the other hand, instead of studying the uniform shear flow as a time-dependent state due to the viscous heating, it is also usual to introduce non-conservative external forces to compensate exactly for the viscous increase of temperature and achieve a steady state. This kind of force is referred to as thermostats and a typical choice of the thermostat force is the friction $-\beta v$ with a constant $\beta\in\mathbb{R}$, see [20, Chapter 3.4]. Inspired by this, we are concerned in this paper with the spatially homogeneous steady problem on the thermostated Boltzmann equation with a deformation force

$$-\beta\nabla_v\cdot(vG_{st})-\alpha\nabla_v\cdot(AvG_{st})=Q(G_{st},G_{st}). \tag{1.1}$$

Here, the unknown $G_{st}=G_{st}(v)$ denotes the non-negative velocity distribution function of particles with velocity $v\in\mathbb{R}^3$. The matrix $A=(a_{ij})\in M_{3\times 3}(\mathbb{R})$ induces a deformation force $-\alpha Av$ with the strength given by the parameter $\alpha>0$ and the constant $\beta\in\mathbb{R}$ is a parameter standing for the strength of the thermostated force. The nonlinear term $Q(\cdot,\cdot)$ is the collision operator defined as

$$Q(F_1,F_2):=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(\omega,v-v_*)\left[F_1(v_*')F_2(v')-F_1(v_*)F_2(v)\right]d\omega dv_*, \tag{1.2}$$

where we have denoted $v'=v+[(v_*-v)\cdot\omega]\omega$ and $v_*'=v_*-[(v_*-v)\cdot\omega]\omega$ with $\omega\in\mathbb{S}^2$ in terms of the conservation laws $v_*+v=v_*'+v'$ and $|v_*|^2+|v|^2=|v_*'|^2+|v'|^2$. Throughout this paper, we let

$$B(\omega,v-v_*)=|v-v_*|^\gamma B_0(\cos\theta),\quad \cos\theta=\omega\cdot\frac{v-v_*}{|v-v_*|},\quad \omega\in\mathbb{S}^2, \tag{1.3}$$

$$0\le\gamma\le 1,\quad 0\le B_0(\theta)\le C|\cos\theta|. \tag{1.4}$$

This includes the cases of the Maxwell molecule $\gamma=0$ and general hard potentials $0<\gamma\le 1$ under the Grad's angular cutoff assumption.