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Semi-Discrete and Fully Discrete Mixed Finite Element Methods for Maxwell Viscoelastic Model of Wave Propagation

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Abstract. Semi-discrete and fully discrete mixed finite element methods are considered for Maxwell-model-based problems of wave propagation in linear viscoelastic solid. This mixed finite element framework allows the use of a large class of existing mixed conforming finite elements for elasticity in the spatial discretization. In the fully discrete scheme, a Crank-Nicolson scheme is adopted for the approximation of the temporal derivatives of stress and velocity variables. Error estimates of the semi-discrete and fully discrete schemes, as well as an unconditional stability result for the fully discrete scheme, are derived. Numerical experiments are provided to verify the theoretical results.

AMS subject classifications: 65N30, 65M60, 65M12

Key words: Maxwell viscoelastic model, mixed finite element, semi-discrete and fully discrete, error estimate, stability.

1 Introduction

Let $\Omega\subset\mathbb{R}^d$($d$=2 or 3) be a bounded open domain with boundary $\partial\Omega$ and T be a positive constant. Consider the following Maxwell viscoelastic model of wave propagation:

$$\begin{cases} \rho\boldsymbol{u}_{tt}=\mathbf{div}\sigma+\boldsymbol{f}, & (\boldsymbol{x},t)\in\Omega\times[0,T],\\ \sigma+\sigma_t=\mathbb{C}\varepsilon(\boldsymbol{u}_t), & (\boldsymbol{x},t)\in\Omega\times[0,T],\\ \boldsymbol{u}=0, & (\boldsymbol{x},t)\in\partial\Omega\times[0,T],\\ \boldsymbol{u}(\boldsymbol{x},0)=\phi_0(\boldsymbol{x}),\ \ \boldsymbol{u}_t(\boldsymbol{x},0)=\phi_1(\boldsymbol{x}),\ \ \sigma(\boldsymbol{x},0)=\psi_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega. \end{cases} \tag{1.1}$$

Here $\boldsymbol{u}=(u_1,\cdots,u_d)^{\mathrm{T}}$ is the displacement field, $\sigma=(\sigma_{ij})_{d\times d}$ the symmetric stress tensor, $\varepsilon(\boldsymbol{u})=(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathrm{T}})/2$ the strain tensor, and $g_t:=\partial g/\partial t$ and $g_{tt}:=\partial^2 g/\partial t^2$ for any function

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$g(\boldsymbol{x},t)$. $\rho(\boldsymbol{x})$ denotes the mass density, and $\mathbb{C}$ a rank 4 symmetric tensor, with

$$0<\rho_0\leq\rho\leq\rho_1<\infty, \qquad \text{a.e. } x\in\Omega, \tag{1.2a}$$

$$0<M_0\tau:\tau\leq\mathbb{C}^{-1}\tau:\tau\leq M_1\tau:\tau, \quad \forall \text{ symmetric tensor } \tau=(\tau_{ij})_{d\times d}, \quad \text{a.e. } \boldsymbol{x}\in\Omega, \tag{1.2b}$$

where ρ_0, ρ_1, M_0 and M_1 are four positive constants, and

$$\sigma:\tau:=\sum_{i=1}^{d}\sum_{j=1}^{d}\sigma_{ij}\tau_{ij}.$$

Note that $\mathbb{C}\varepsilon(\boldsymbol{u}_t)$ is of the form

$$\mathbb{C}\varepsilon(\boldsymbol{u}_t)=2\mu\varepsilon(\boldsymbol{u}_t)+\lambda\mathrm{div}\boldsymbol{u}_t I \tag{1.3}$$

for an isotropic elastic medium, where μ, λ are the Lamé parameters, and I the identity matrix. $\boldsymbol{f}=(f_1,\cdots,f_d)^{\mathrm{T}}$ is the body force, and $\phi_0(\boldsymbol{x})$, $\phi_1(\boldsymbol{x})$, $\psi_0(\boldsymbol{x})$ are initial data.

Numerous materials simultaneously display elastic and viscous kinematic behaviours. Such a feature, called viscoelasticity, can be characterized by using springs, which obey the Hooke's law, and viscous dashpots, which obey the Newton's law. Different combinations of the springs and dashpots lead to various viscoelastic models, e.g., the three classical models of Zener, Voigt and Maxwell. We note that there is a unified framework to describe the general constitutive law of viscoelasticity by using convolution integrals in time with some kernels [8,11,29]; however, the integral forms of constitutive laws, compared with the differential forms, bring more difficulties to the design of algorithms due to the numerical convolution integral. We refer the reader to [5,9–13,30–32] for several monographs on the development and applications of viscoelasticity theory.

The numerical simulation of wave propagation in viscoelastic materials was first discussed by Kosloff et al. in [20, 21], where memory variables were introduced to avoid the convolutional integral in the constitutive relation. Later, finite difference methods were developed in [6, 28, 35] for the model with memory variables. There are considerable research efforts on the finite element discretization in this field. In [18] Janovsky et al. studied the continuous/discontinuous Galerkin finite element discretization and used a numerical quadrature formula to approximate the Volterra time integral term. Ha et al. [14] proposed a nonconforming finite element method for a viscoelastic complex model in the space–frequency domain. Bécache el at. [3] applied a family of mixed finite elements with mass lumping, together with a leap-frog scheme in time discretization, to numerically solve the Zener model, and showed that their scheme is stable under certain CFL condition. In [24–26], Rivière et al. analyzed discontinuous Galerkin methods with a Crank-Nicolson temporal discrete scheme for quasistatic linear viscoelasticity and linear/nonlinear diffusion viscoelastic models. Rognes and Winther [27] proposed mixed finite element methods for quasistatic Maxwell and Voigt models using weak symmetry, and used a second backward difference scheme in the full discretization. Lee [22] studied mixed finite element methods with weak symmetry for the Zener, Voigt and