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A SECOND ORDER UNCONDITIONALLY CONVERGENT FINITE ELEMENT METHOD FOR THE THERMAL EQUATION WITH JOULE HEATING PROBLEM*

Xiaonian Long
College of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450045, China
Email: longxiaonian@lsec.cc.ac.cn
Qianqian Ding[1)]
School of Mathematics, Shandong University, Jinan 250100, China
Email: dingqianqian@lsec.cc.ac.cn, qqding@sdu.edu.cn

Abstract

In this paper, we study the finite element approximation for nonlinear thermal equation. Because the nonlinearity of the equation, our theoretical analysis is based on the error of temporal and spatial discretization. We consider a fully discrete second order backward difference formula based on a finite element method to approximate the temperature and electric potential, and establish optimal L^2 error estimates for the fully discrete finite element solution without any restriction on the time-step size. The discrete solution is bounded in infinite norm. Finally, several numerical examples are presented to demonstrate the accuracy and efficiency of the proposed method.

Mathematics subject classification: 65M12, 65M15, 65M60, 35K61.
Key words: Thermal equation, Joule heating, Finite element method, Unconditional convergence, Second order backward difference formula, Optimal L^2-estimate.

1. Introduction

In this work, we consider the following nonlinear heat equation:

$$\theta_t - \Delta\theta = \sigma(\theta)|\nabla\phi|^2 \quad \text{in} \quad Q_T, \tag{1.1}$$

$$-\nabla\cdot(\sigma(\theta)\nabla\phi) = 0 \quad \text{in} \quad Q_T, \tag{1.2}$$

where $Q_T = \Omega\times(0,T)$, $T>0$ is a given finite final time and Ω is a bounded domain with Lipschitz boundary $\partial\Omega$. The unknown variables are the electric potential ϕ and the temperature θ. The physical parameter is the electrical conductivity σ, which depends on the temperature θ. The Joule heating is $\sigma(\theta)|\nabla\phi|^2$. The system is considered along with the following initial and boundary conditions:

$$\theta(x,0) = \theta^0 \qquad\qquad \forall\, x\in\Omega, \tag{1.3}$$

$$\theta(x,t) = 0, \quad \phi(x,t) = g \quad \text{on} \quad S_T, \tag{1.4}$$

where $S_T = \partial\Omega\times(0,T)$. Following the previous works [9,26], we assume that $\sigma\in W^{1,\infty}(\mathbb{R})$ and

$$\sigma_1 \le \sigma(s) \le \sigma_2$$

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1) Corresponding author

for some positive constants σ_1 and σ_2.

The thermistor problem is a coupled system of nonlinear PDEs, which consists of the heat equation with the Joule heating as a source, and the current conservation equation with temperature dependent electrical conductivity. It is known that Joule heating (also known as Ohmic heating and resistive heating) will be produced when the existing electric current acts on a conductive liquid, which may be the main heat source in many real applications.

Many authors have discussed the numerical methods for the time-dependent nonlinear thermistor equations. We recall some important studies about the problem. Yue [22] used the piecewise linear finite element approximation to solve the heat equations and to show the L^2-error estimate and H^1-error estimate all h-order. Elliott et al [9] proved an optimal L^2-error estimate with the condition $\Delta t = \mathcal{O}(h^{1/2})$ in 3-dimensional space, in which a linearized semi-implicit Euler scheme with a linear Galerkin finite element method was used. A finite element approach was introduced in [14,15], in which the error estimates of a linearized backward Euler Galerkin method for a porous media flow and the thermistor system were obtained, respectively, under the condition of h and Δt being smaller than a positive constant. Other related studies on this topic can be found in [1–3, 8, 19–21, 27] and the references therein. In all these works, error estimates were established under certain time-step restrictions, which depend upon the dimension, the scheme, and the nonlinearity of the equations in general.

We also make some investigations on the optimal error estimates of the unconditional convergence for the nonlinear heat equation. In [13], the authors proved the optimal error estimates of the fully discrete Crank-Nicolson Galerkin method unconditionally, in which they obtained the L^2 optimal error for temperature and the optimal $L^{12/5}$ error estimate for the electric potential. Let us note that here and in what follows, the term unconditional means that the convergence of numerical solution does not depend on CFL condition. In a two-dimensional nonconvex polygon, the time-dependent nonlinear thermistor problem was studied in [10], in which the authors proved the optimal error estimates of a linearized backward Euler Galerkin finite element method unconditional. However, the motivation of our study is to give the optimal L^2 error estimate for the temperature and electric potential based on second order backward difference formula (BDF2).

In this work, we discretize the problem (1.1)-(1.4) by the second order backward difference formula in temporal domain and by the mixed finite element in spatial domain. More precisely, the stable mixed finite elements are used to approximate temperature field and electric potential. In order to simplify the calculation in practice, we treat the nonlinear Joule heating term by explicit scheme. Thus the fully discrete scheme proposed here requires only solving a linear system per time step. In order to get rid of certain restrictions like $\Delta t \leq Ch^s$ on the time step, we employ a error splitting approach in terms of the time-discrete system. We unconditionally derive optimal error estimates for temperature and electric potential. In addition, several numerical examples are implemented to demonstrate both accuracy and efficiency of the discrete scheme.

For the mathematical setting of the heat equations (1.1)-(1.2) with the initials and boundary conditions (1.3)-(1.4), we introduce some function spaces and their associated norms. For all $m \in \mathbb{N}^+$, $1 \leq p \leq \infty$, let $W^{m,p}(\Omega)$ denote the standard Sobolev space and it is written as $H^m(\Omega)$ when $p = 2$. The norm in $W^{m,p}(\Omega)$, denoted by $\|\cdot\|_{m,p}$, is defined as follows:

$$\|v\|_{m,p} = \Bigg(\sum_{|\gamma|\leq m} \|D^\gamma v\|_{0,p}^p\Bigg)^{1/p} \quad \text{with} \quad \|v\|_{0,p} = \bigg(\int_\Omega |v|^p \, dx\bigg)^{1/p}, \quad 1 \leq p < \infty,$$