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Shift-Splitting Iteration Method and Its Variants for Solving Continuous Sylvester Equations

Xu Li* and Ning He

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou 730050, P.R. China.

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Abstract. A shift-splitting iteration method for solving large sparse continuous Sylvester equations is developed. This single-step iteration algorithm demonstrates a better computational efficiency than the previously used two-step iterative methods. We also propose two variants — viz. inexact and accelerated shift-splitting iteration methods. The convergence properties of all algorithms are studied and the quasi-optimal iteration parameter of shift-splitting is derived. Numerical examples demonstrate the efficiencies of the three methods, especially for equations with ill-conditioned coefficient matrices.

AMS subject classifications: 15A24, 15A30, 15A69, 65F10, 65F30, 65F50, 65H10

Key words: Continuous Sylvester equation, shift-splitting iteration, inexact iteration, convergence.

1. Introduction

The large and sparse continuous Sylvester equation has the form

$$AX + XB = C, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ are positive semi-definite complex matrices. Positive definiteness of one or both of the coefficient matrices guarantees the existence and uniqueness of the solution of (1.1) [39, 41]. Such continuous Sylvester equations arise in various applications, including control and system theory [40, 47, 53], linear system stability [31], linear algebra [28], signal processing [1], image restoration [16], and filtering [30, 32].

Let I_k denote the identity matrix of order k. The Eq. (1.1) can be written as the linear system

$$\mathbf{A}x = c, \tag{1.2}$$

where $\mathbf{A} = I_n \otimes A + B^{\mathrm{T}} \otimes I_m$, $\otimes$ is the Kronecker product, B^{T} is the transpose of B, and x and c are the vectors consisting of the concatenated columns of the matrices X and C,

*Corresponding author. *Email addresses:* `lixu@lut.edu.cn` (X. Li), `17691159643@163.com` (N. He)

respectively. The system (1.2) can be used in theoretical analysis, but solving (1.2) is a cost consuming process.

There are various numerical methods for solving the Eq. (1.1). In particular, for small coefficient matrices one can engage direct solvers such as Bartels-Stewart [14] and Hessenberg-Schur [27] methods. In the case of large and sparse coefficient matrices, one usually uses an iterative scheme — e.g. Smith's method [51], alternating direction implicit (ADI) method [16, 26, 33, 42, 54], gradient based algorithms [46, 55]. For other methods, the readers can consult [25,29,36,52] (see [50] and references therein for a detailed survey).

The Hermitian and skew-Hermitian splitting (HSS) iteration method proposed by Bai *et al.* [8] in order to solve non-Hermitian positive definite linear systems, was further developed by various authors — cf. Refs. [3, 4, 6, 7, 9, 12, 15, 45, 57, 59, 60]. The study of HSS method for the Sylvester equation (1.1) was started in [2] and resulted in variety of algorithms, including preconditioned HSS (PHSS) [43], normal and skew-Hermitian splitting (NSS) [63], positive-definite and skew-Hermitian splitting (PSS) [56], preconditioned PSS (PPSS) [66], generalized HSS (GHSS) [65], parameterized HSS [44], modified HSS (MHSS) [64] and others [37,38].

Recently, Bai *et al.* [13] introduced a shift-splitting (SS) iteration method for non-Hermitian positive definite linear systems. Subsequently, the SS method was extended to linear systems with special structure, such as standard saddle point problems [19,62], non-symmetric saddle point problems [20, 21, 35, 67], singular saddle point problems [21, 49], generalized saddle point problems [48, 49], block 3 × 3 saddle point problems [18], elasticity problems [22], and time-harmonic eddy current problems [17,23].

In this paper, we apply the SS algorithm to the Eq. (1.1). In addition, we present an inexact SS (ISS) method and an accelerated variant of the SS (ASS) method. The convergence properties of the proposed methods are also discussed.

The rest of this paper is structured as follows. The SS algorithm and its convergence are discussed in Sections 2 and 3, respectively. Section 4 describes the ISS method and analyzes its convergence. In Section 5, the ASS algorithm and its convergence are presented. The results of numerical experiments shown Section 6, demonstrate the efficiencies of the algorithms proposed. Lastly, Section 7 concludes this paper.

2. Shift-Splitting Iteration Method

The shift-splitting of A and B is defined by

$$A = \frac{1}{2}(\alpha I_m + A) - \frac{1}{2}(\alpha I_m - A), \tag{2.1}$$

$$B = \frac{1}{2}(\beta I_n + B) - \frac{1}{2}(\beta I_n - B), \tag{2.2}$$

where $\alpha > 0$ and $\beta > 0$ are constants [13]. Using (2.1)-(2.1), we represent the continuous Sylvester equation (1.1) in the form

$$(\alpha I_m + A)X + X(\beta I_n + B) = (\alpha I_m - A)X + X(\beta I_n - B) + 2C. \tag{2.3}$$