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Self-Similar Solutions of Leray's Type for Compressible Navier-Stokes Equations in Two Dimension

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Abstract. We study the backward self-similar solution of Leray's type for compressible Navier-Stokes equations in dimension two. The existence of weak solutions is established via a compactness argument with the help of an higher integrability of density. Moreover, if the density belongs to $L^\infty(\mathbb{R}^2)$ and the velocity belongs to $L^2(\mathbb{R}^2)$, the solution is trivial; that is $(\rho,\mathbf{u})=0$.

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Key words: Navier-Stokes equations, self-similar solutions, compressible.

1 Introduction

This work concerns the backward self-similar solution of Leray's type for compressible isentropic Navier-Stokes equations with a positive number M in $\mathbb{R}^n$ as the adiabatic constant $\gamma>\frac{n}{2}$

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0,\\ \partial_t(\rho\mathbf{u})+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})-\mu\Delta\mathbf{u}-\xi\nabla\operatorname{div}\mathbf{u}+M\nabla\rho^\gamma=0,\\ (\rho,\mathbf{u})|_{t=0}=(\rho_0,\mathbf{u}_0),\end{cases}\tag{1.1}$$

where $\mu>0$, $\xi\in\mathbb{R}$ with $\mu+\xi>0$, ρ and $\mathbf{u}$ stand for the density and velocity of the flow, respectively. The positive constant M in (1.1) is the squared inverse of

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the Mach number. The weak solution of (1.1) in $\mathbb{R}^n$ had been established first by Lions [15] for $\gamma\geq\frac{3}{2}$ as $n=2$ and $\gamma\geq\frac{9}{5}$ as $n=3$, and later extended by Feireisl *et al.* [6] to allow $\gamma>\frac{n}{2}$ in the Leray sense, see also [21] for $\gamma=1$ in dimension two and [1] for the general pressure in this direction.

Definition 1.1. *A couple* $(\rho,\mathbf{u})$ *with* $\rho\in L^\infty(0,T;L^\gamma(\mathbb{R}^n))$ *and* $\nabla\mathbf{u}\in L^2((0,T)\times\mathbb{R}^n)$ *is said to be a renormalized weak solution to* (1.1) *with finite energy if* $(\rho,\mathbf{u})$ *satisfies the following:*

- *The kinetic energy is bounded, i.e.,* $\rho|\mathbf{u}|^2\in L^\infty(0,T;L^1(\mathbb{R}^n))$. *The density function is nonnegative, i.e.,* $\rho\geq 0$.
- *The momentum equation holds true in the sense of distributions.*
- *The identity*
$$\partial_t b(\rho)+\operatorname{div}\big(\mathbf{u}b(\rho)\big)=\big(b(\rho)-b'(\rho)\rho\big)\operatorname{div}\mathbf{u} \tag{1.2}$$
holds true in the sense of distributions for any function $b(t)\in C([0,\infty)\cap C^1(\mathbb{R}^+)$ *such that* $b(t)-b'(t)t\in C([0,\infty))$ *is bounded on* $[0,\infty)$.
- *The global energy inequality*
$$E\big(\rho(t),\mathbf{u}(t)\big)+\int_0^t\int_{\mathbb{R}^n}\Big(\mu|\nabla\mathbf{u}|^2+\xi|\operatorname{div}\mathbf{u}|^2\Big)dxds\leq E(\rho_0,\mathbf{u}_0) \tag{1.3}$$
holds true with
$$E(\rho,\mathbf{u})=\begin{cases}\dfrac{1}{2}\|\rho|\mathbf{u}|^2\|_{L^1(\mathbb{R}^n)}+\dfrac{M}{\gamma-1}\displaystyle\int_{\mathbb{R}^n}\rho^\gamma dx, & \text{if}\quad \gamma>1,\\ \dfrac{1}{2}\|\rho|\mathbf{u}|^2\|_{L^1(\mathbb{R}^n)}+M\displaystyle\int_{\mathbb{R}^n}\rho\ln(1+\rho)dx & \text{if}\quad \gamma=1.\end{cases}$$

It is worth noticing that a number of efforts have been paid on the existence of weak solutions for both the time-discretized and the steady counterparts of (1.1) as $\gamma\in[1,\frac{n}{2}]$, see for instance [4,12,13,19]. For the time-discretized version of (1.1), the concentration-cancellation occurs in dimension two (see [15, Section 6.6]) and in dimension three as $\gamma>1$ (see [20]); while the concentration set has been verified to be $(\mathcal{H}^1,1)$ rectifiable in dimension three as $\gamma=1$ in [20], where $\mathcal{H}^1$ stands for the one dimensional Hausdorff measure. For the steady version of (1.1) as $\gamma=1$, the concentration cancellation in dimension two has been verified in the work [7] due to a potential type estimate of the density (see [7, Lemma 3.1]), also see [14,17,22,23] for the most recent improvement along this direction in dimension three. For time-dependent version (1.1), in [10] the author considered the