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Several Optimal Bounds for Some Means Derived From the Lemniscatic Mean

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Abstract. In this paper, we present sharp bounds for some bivariate means derived from the lemniscatic mean by Neuman, in terms of the harmonic, arithmetic and contraharmonic means.

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Key words: Lemniscate mean; bivariate means; bound; inequalities.

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1 Introduction

For $a,b>0$ with $a\neq b$, the lemniscate mean LM (see [3, (2.7)] and [2, P. 259]) is defined as follows:

$$LM(a,b)=\begin{cases} \dfrac{\sqrt{a^2-b^2}}{\left(\operatorname{arcsl}\sqrt[4]{1-\frac{b^2}{a^2}}\right)^2}, & a>b,\\ \dfrac{\sqrt{b^2-a^2}}{\left(\operatorname{arcslh}\sqrt[4]{\frac{b^2}{a^2}-1}\right)^2}, & a<b,\\ a, & a=b, \end{cases} \tag{1.1}$$

where

$$\operatorname{arcsl}x=\int_0^x \frac{dt}{\sqrt{1-t^4}}, \quad |x|\leq 1 \tag{1.2}$$

and

$$\operatorname{arcslh}x=\int_0^x \frac{dt}{\sqrt{1+t^4}}, \quad x\in\mathbb{R} \tag{1.3}$$

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are Gauss arc lemniscate sine and the hyperbolic arc lemniscate sine functions respectively (see [23, Ch.1]). Following Neuman [19, Proposition 3.1], another pair of the arc lemniscate functions, Gauss arc lemniscate tangent function and the hyperbolic arc lemniscate tangent function arctlh are defined by

$$\operatorname{arctl} x = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), \quad x \in \mathbb{R}, \tag{1.4}$$

and

$$\operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), \quad |x| < 1, \tag{1.5}$$

respectively.

The limiting values of the above four functions are (see [21, 19.20.2], [20])

$$\begin{aligned} \omega &= \operatorname{arcsl} 1 = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{2\pi}} = 1.31103\cdots, \\ \kappa &= \operatorname{arcslh}(+\infty) = \sqrt{2}\omega = 1.85407\cdots, \\ \sigma &= \operatorname{arcslh} 1 = \frac{\omega}{\sqrt{2}} = 0.92703\cdots, \\ \tau &= \operatorname{arctl} 1 = \operatorname{arcsl}\left(\frac{1}{\sqrt[4]{2}}\right) = 0.89558\cdots \end{aligned} \tag{1.6}$$

where $\Gamma(x)$ is the classical Euler gamma function.

For more information on the arc lemniscate functions, the reader may see references [1,4,5,8,13,18,24–26].

Let

$$G = \sqrt{ab}, \quad A = \frac{a+b}{2}, \quad Q = \sqrt{\frac{a^2+b^2}{2}}, \tag{1.7}$$

be the geometric, arithmetic, and quadratic means of two distinct positive real numbers a and b, respectively. The means derived from the lemniscatic mean are defined by Neuman as follows [19, (6.4)]:

$$\begin{aligned} LM_{GA} &= LM_{GA}(a,b) = LM(G,A), \\ LM_{AG} &= LM_{AG}(a,b) = LM(A,G), \\ LM_{AQ} &= LM_{AQ}(a,b) = LM(A,Q), \\ LM_{QA} &= LM_{QA}(a,b) = LM(Q,A). \end{aligned} \tag{1.8}$$

Other means used in this paper are the harmonic mean H and the contraharmonic mean C which are defined as follows

$$H = \frac{2ab}{a+b}, \quad C = \frac{a^2+b^2}{a+b}. \tag{1.9}$$