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Integral Equation Method for a Non-Selfadjoint Steklov Eigenvalue Problem

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Abstract. We propose a numerical method for a non-selfadjoint Steklov eigenvalue problem of the Helmholtz equation. The problem is formulated using boundary integrals. The Nyström method is employed to discretize the integral operators, which leads to a non-Hermitian generalized matrix eigenvalue problems. The spectral indicator method (SIM) is then applied to calculate the (complex) eigenvalues. The convergence is proved using the spectral approximation theory for (non-selfadjoint) compact operators. Numerical examples are presented for validation.

AMS subject classifications: 65N25, 35P15, 45C05, 35P25

Key words: Steklov eigenvalues, non-selfadjoint problems, integral equations, Nyström method, spectral projection.

1 Introduction

In mathematical physics, eigenvalue problems with the spectral parameters in the boundary conditions are often called the Steklov eigenvalue problems [17]. They have wide applications in surface waves, mechanical oscillators immersed in a viscous fluid, the vibration modes of a structure in contact with an incompressible fluid, inverse scattering theory, etc [4, 6, 7, 17].

There exist many works on the computation of Steklov eigenvalues, in particular, the finite element methods (see, e.g., [1–3, 19, 23, 27, 30]). In recent years, the non-selfadjoint Steklov eigenvalue problem for the Helmholtz equation received increased attention in the numerical analysis community due to their applications in the inverse scattering theory [8, 11, 18, 21, 28, 29, 31].

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In this paper, we consider the computation of the Steklov eigenvalues for the Helmholtz equations with a constant refractive index in $\mathbb{R}^2$. The problem is reformulated using boundary integrals. The Nyström method is employed for discretization. The convergence is proved following [9], which is based on the spectral approximation theory for (non-selfadjoint) compact operators [24]. Since the discretization is with respect to the boundary of the domain, the computational complexity is lower than the finite element methods. The integral formulation of the Steklov eigenvalue problem has an additional advantage. It leads to a linear eigenvalue problem. In contrast, the integral formulation for the Dirichlet eigenvalue problem lead to a nonlinear eigenvalue problem [25].

Due to the non-selfadjoint nature of the problem, the Nyström method leads to a generalized eigenvalue problems of dense non-Hermitian matrices, which are highly challenging for classical methods in numerical linear algebra. We resort to a new eigensolver, called the spectral indicator method (SIM) [12–14, 32]. Using the spectral projection, SIM computes an indicator for a region S on the complex plane and decides if S contains eigenvalue(s). A region containing eigenvalue(s) is subdivided. The procedure iterates until the eigenvalues are located up to a given precision.

The rest of the paper is arranged as follows. The boundary integral formulation for the Steklov eigenvalue problem is derived in Section 2. In Section 3, the Nyström method is employed to discretize the integrals. The properties of the discrete operators are established and the convergence is proved using the spectral approximation theory for (non-selfadjoint) compact operators. Numerical examples validating the proposed method are presented in Section 4.

2 Steklov eigenvalue problem

Let $\Omega\subset\mathbb{R}^2$ be an open bounded domain with $C^{2,1}$ boundary. Denote by $\mu\in\mathbb{C}$ the constant index of refraction and $\kappa\in\mathbb{R}^+$ the wavenumber. We consider the Steklov eigenvalue problem of finding $\lambda\in\mathbb{C}$ and non-trivial function $u\in H^1(\Omega)$ such that

$$\begin{aligned} \Delta u+\kappa^2\mu u=0 \quad & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}+\lambda u=0 \quad & \text{on } \Gamma, \end{aligned} \tag{2.1}$$

where $\Gamma:=\partial\Omega$ and ν is the unit outward normal to Γ. The problem is non-selfadjoint if μ is complex.

Let Φ_κ be the Greens function given by

$$\Phi_k(x,y):=\frac{\mathrm{i}}{4}\mathrm{H}_0^{(1)}(\kappa|x-y|) \quad \text{for } x,y\in\mathbb{R}^2,$$

where $\mathrm{H}_0^{(1)}$ is the Hankel function of the first kind of order zero. The single and double layer potentials are defined respectively as

$$(\mathcal{SL}_\kappa[\phi])(x):=\int_\Gamma \Phi_k(x,y)\phi(y)\mathrm{d}s(y), \quad x\in\mathbb{R}^2\backslash\Gamma$$