*Ann. of Appl. Math.*
**34**:2(2018), 165-177

# NEW DYNAMIC INEQUALITIES FOR DECREASING FUNCTIONS AND THEOREMS OF HIGHER INTEGRABILITY*

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**Abstract**

In this paper we establish some new dynamic inequalities on time scales which contain in particular generalizations of integral and discrete inequalities due to Hardy, Littlewood, Pólya, D'Apuzzo, Sbordone and Popoli. We also apply these inequalities to prove a higher integrability theorem for decreasing functions on time scales.

**Keywords** reverse Hölder's inequality; Gehring class; higher integrability; Hardy-Littlewood-Pólya inequality; time scales

**2000 Mathematics Subject Classification** 26D15; 34A40; 34N05; 39A12

## 1 Introduction

The classical integral Hölder inequality states that if $f$ and $g$ are positive measurable functions defined on $\mathbb{I}$, $p > 1$, $q > 1$ and $1/p + 1/q = 1$, then

$$\int_{\mathbb{I}} |f(x)g(x)|\mathrm{d}x \leq \left[\int_{\mathbb{I}} |f(x)|^p \mathrm{d}x\right]^{\frac{1}{p}} \left[\int_{\mathbb{I}} |g(x)|^q \mathrm{d}x\right]^{\frac{1}{q}}. \tag{1.1}$$

For generalizations and extensions in the literature we refer the reader to the papers [6–9, 12–15]. In this paper we establish some new dynamic inequalities on time scales. For related dynamic inequalities on time scales, we refer the reader to the books [1, 2] and the papers [16, 17].

The paper is organized as follows. In Section 2 we give a brief overview on time scales. In Section 3, we prove some new dynamic inequalities via convexity functions which contain inequalities given by Hardy, Littlewood and Pólya. We also establish

*Manuscript received July 6, 2017; Revised March 26, 2018

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some new dynamic inequalities on time scales for decreasing functions. These inequalities contain in particular generalizations of integral inequalities due to Hardy, D'Apuzzo and Sbordone and Popoli. We apply also these inequalities to prove a higher integrability theorem for monotone decreasing function on time scales.

## 2 Preliminaries on Time Scales

For a good introduction on time scales we refer the reader to the book [4]. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ respectively. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided $f$ is continuous at right-dense points and at left-dense points in $\mathbb{T}$, whose left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function $\mu$ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$, $f(\sigma(t))$ is defined by $f^\sigma(t)$. We now recall the product and quotient rules for the derivative of the product $fg$ and the quotient $f/g$ (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable functions $f$ and $g$

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \quad \text{and} \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{2.1}$$

One chain rule is

$$(f^\gamma(t))^\Delta = \gamma \int_0^1 [hf^\sigma + (1-h)f]^{\gamma-1} \, \mathrm{d}h f^\Delta(t), \tag{2.2}$$

which is a simple consequence of Keller's chain rule [4, Theorem 1.90]. Another chain rule is as follows: Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and

$$f^\Delta(g(t)) = f'(g(d))g^\Delta(t), \quad \text{for } d \in [t, \sigma(t)]. \tag{2.3}$$

In this paper we will consider the (delta) integral which we can be defined as follows. If $F^\Delta(t) = f(t)$, then the Cauchy (delta) integral of $f$ is defined by $\int_{t_0}^t f(s)\Delta s := F(t) - F(t_0)$. It can be shown (see [4]) that if $f \in C_{rd}(\mathbb{T})$, then the Cauchy integral $F(t) := \int_{t_0}^t f(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $F^\Delta(t) = f(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^\infty f(t)\Delta t = \lim\limits_{b\to\infty} \int_a^b f(t)\Delta t$. Integration on discrete time scales is defined by

$$\int_a^b f(t)\triangle t = \sum_{t\in[a,b)} \mu(t)f(t).$$

The integration by parts formula on time scales is given by