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SEMICLASSICAL LIMIT TO THE GENERALIZED NONLINEAR SCHRÖDINGER EQUATION*†

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Abstract

In this paper, we investigate the semiclassical limit of the generalized nonlinear Schrödinger equation for initial data with Sobolev regularity. Also, we will analyze the structure of the fluid dynamical system with quantum effect corresponding to the semiclassical limit of the generalized nonlinear Schrödinger equation.

Keywords quantum hydrodynamics; dispersive limit; compressible Euler equation

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1 Introduction

Hydrodynamics equations with quantum effect describe the hydrodynamical properties and states of some important physical phenomena such as semiconductor, superconductor and superflow. This kind of equations have theoretical significance and practical value. From the semiclassical limit of the nonlinear Schrodinger (NLS) equation with Plank constant h, we can derive various hydrodynamics equations with quantum effect when $h \to 0$.

It is well known that the quantum hydrodynamics equations (QHD) can be derived based on the moment method, which is analogous to the derivation of the compressible Euler equation from the Boltzmann equation by taking the zeroth, first and second order velocity moments of the quantum Boltzmann equation and

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resulting in a hydrodynamical model which then has to be closed in an approximate way, that is, a reasonable macroscopic approximation for the quantum heat flow tensor has to be derived by using additional (quantum) physical properties of the particle ensembles. Moreover, in the case of high electric fields, small mean-free-path asymptotics have been used to derive QHD-models.

When the time and distance scales are large enough relative to the Plank constant h, the system will approximately obey the laws of classical, Newtonian mechanics. That is, quantum mechanics becomes Newtonian mechanics as $h \to 0$. The asymptotics of quantum variables as $h \to 0$ are known as semiclassical expressing this limiting behavior.

In the semiclassical limit or WKB limit and when ∇_x and ∂_t scale like ϵ as $\epsilon \to 0$ (ϵ is the scaled Planck constant), the quantum-mechanical pressure becomes negligible. The isentropic compressible Euler equation can be formally recovered from the nonlinear Schrödinger equation in this limit. This fact was proven rigorously by Jin, Levermore and McLaughlin [5, 6] for the one-dimensional integrable case using the inverse scattering technique and by Grenier [3] for higher dimensions in situations where no vortices are involved.

Very similar model equations have been used for quite a while in other areas of theoretical and computational physics, for instance, in superfluidity [11, 12] and in superconductivity [2].

2 Semiclassical Limit to the Nonlinear Schrödinger Equation in Short Time Range

In this section, we consider the following nonlinear Schrödinger (NLS) equation with rapidly oscillating data

$$ih\partial_t\psi_h + \frac{h^2}{2}\Delta_x\psi_h + f(|\psi_h|^2)\psi_h = 0, \tag{2.1}$$

$$\psi_h(0, x) = a^0(x, h)\exp\Big(\frac{iS^0(x)}{h}\Big), \tag{2.2}$$

where $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$, $S^0(x) \in H^s(\mathbb{R}^d)$ for s large enough. And a^0 is a function, polynomial in h with coefficients of Sobolev regularity in x. h is the Plank constant and ψ_h is the wave function.

We will study the semiclassical limit of equation (2.1)-(2.2) and determine the limiting dynamics of any function of the field ψ_h as $h \to 0$.

Remark 2.1 When $f(x) = x$, equation (2.1) appears in the phenomenological description of superfluidity of an almost ideal Bose gas [10]. In this case, the squared modulus of the wave function $\psi\bar{\psi}$ is interpreted as the particle number density in the