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$\mathbb{Z}_3$-CONNECTIVITY OF 4-EDGE-CONNECTED TRIANGULAR GRAPHS*†

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Abstract

A graph G is k-triangular if each of its edge is contained in at least k triangles. It is conjectured that every 4-edge-connected triangular graph admits a nowhere-zero 3-flow. A triangle-path in a graph G is a sequence of distinct triangles $T_1T_2\cdots T_k$ in G such that for $1 \le i \le k-1$, $|E(T_i) \cap E(T_{i+1})| = 1$ and $E(T_i) \cap E(T_j) = \emptyset$ if $j > i+1$. Two edges $e, e' \in E(G)$ are triangularly connected if there is a triangle-path $T_1, T_2, \cdots, T_k$ in G such that $e \in E(T_1)$ and $e' \in E(T_k)$. Two edges $e, e' \in E(G)$ are equivalent if they are the same, parallel or triangularly connected. It is easy to see that this is an equivalent relation. Each equivalent class is called a triangularly connected component. In this paper, we prove that every 4-edge-connected triangular graph G is $\mathbb{Z}_3$-connected, unless it has a triangularly connected component which is not $\mathbb{Z}_3$-connected but admits a nowhere-zero 3-flow.

Keywords $\mathbb{Z}_3$-connected; nowhere-zero 3-flow; triangular graphs
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1 Introduction

We follow the notations and terminology of [1] except otherwise stated.

The concept of k-triangular graphs was introduced by Broersma and Veldman in [2]. A graph is k-triangular if each of its edge is contained in at least k triangles. A 1-triangular graph is also referred to as a triangular graph.

A *triangle-path* in a graph G is a sequence of distinct triangles $T_1T_2\cdots T_k$ in G such that for $1 \le i \le k-1$, $|E(T_i) \cap E(T_{i+1})| = 1$ and $E(T_i) \cap E(T_j) = \emptyset$ if $j > i+1$. Two edges $e, e' \in E(G)$ are *triangularly connected* if there is a triangle-path $T_1, T_2, \cdots, T_k$ in G such that $e \in E(T_1)$ and $e' \in E(T_k)$. Two edges $e, e' \in E(G)$ are *equivalent* if they are the same, parallel or triangularly connected. It is easy to see that this is an equivalent relation. Each equivalent class is called a *triangularly connected component*. A graph is *triangularly connected* if and only if it has only one triangularly connected component.

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Let A be a nontrivial additive Abelian group and $A^* = A - \{0\}$. Let G be a graph with an arbitrary orientation. For any $v \in V(G)$, we denote the set of arcs with tails at v by $E^+(v)$ and heads at v by $E^-(v)$. Following the definitions in [6], we give

$$F(G, A) = \{f | \, f : E(G) \to A\} \quad \text{and} \quad F^*(G, A) = \{f | \, f : E(G) \to A^*\}.$$

For each $f \in F(G, A)$, the boundary of f is a function $\partial f : V(G) \to A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where the symbol "$\sum$" refers to the addition in A. We define

$$Z(G, A) = \Big\{ b | \, b : V(G) \to A \text{ with } \sum_{v \in V(G)} b(v) = 0 \Big\}.$$

An A-*nowhere-zero flow* (abbreviated as A-NZF) in G is a function $f \in F^*(G, A)$ such that $\partial f \equiv 0$. For any $b \in Z(G, A)$, a function $f \in F^*(G, A)$, with $\partial f = b$ is called an (A, b)-NZF. A graph G is A-*connected*, if for every $b \in Z(G, A)$, there exists an (A, b)-NZF. For an Abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected. It has been observed that the A-connectivity of G is independent of the orientation of G, so we usually give G an arbitrary orientation.

The nowhere-zero flow problems are introduced by Tutte [11] and surveyed by Jaeger [7] and Zhang [13]. The following three conjectures are proposed by Tutte.

Conjecture 1.1 Every bridgeless graph admits a $\mathbb{Z}_5$-NZF.

Conjecture 1.2 Every bridgeless graph without a Peterson minor admits a $\mathbb{Z}_4$-NZF.

Conjecture 1.3 Every 4-edge-connected graph admits a $\mathbb{Z}_3$-NZF.

These three problems remain open today. As a generalization of A-NZF, Jaeger [8] introduced the concept of A-connectivity and proposed the following conjecture:

Conjecture 1.4 Every 5-edge-connected graph is $\mathbb{Z}_3$-connected.

For triangular graphs, Xu and Zhang [12] proposed a weaker version of Conjecture 1.3:

Conjecture 1.5 Every 4-edge-connected triangular graph has a $\mathbb{Z}_3$-NZF.

It was further asked (Problem 1 in [7]) whether every 4-edge-connected triangular graph is $\mathbb{Z}_3$-connected. This was shown in the negative in [7].

In [4], Hou et al. proved that every 4-edge-connected 2-triangular graph is $\mathbb{Z}_3$-connected, and further they pointed out that 2-triangularity is best possible and a class of 4-edge-connected triangular graphs which are not $\mathbb{Z}_3$-connected was constructed. In these counterexamples, each of their triangularly connected component has at least one vertex of degree 2.