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LIMIT CYCLES OF THE GENERALIZED POLYNOMIAL LIÉNARD DIFFERENTIAL SYSTEMS*

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Abstract

Using the averaging theory of first and second order we study the maximum number of limit cycles of generalized Liénard differential systems

$$\begin{cases} \dot{x} = y + \epsilon h_l^1(x) + \epsilon^2 h_l^2(x), \\ \dot{y} = -x - \epsilon(f_n^1(x)y^{2p+1} + g_m^1(x)) + \epsilon^2(f_n^2(x)y^{2p+1} + g_m^2(x)), \end{cases}$$

which bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, where ϵ is a small parameter. The polynomials h_l^1 and h_l^2 have degree l; f_n^1 and f_n^2 have degree n; and g_m^1, g_m^2 have degree m. $p \in \mathbb{N}$ and $[\cdot]$ denotes the integer part function.

Keywords limit cycle; periodic orbit; Liénard differential system; averaging theory

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1 Introduction and Statement of the Main Results

One of the main problems in the theory of differential systems is the study of the existence, number and stability of limit cycles. A limit cycle of a differential system is an isolated periodic orbit in the set of all periodic orbits of the differential system. These last years hundreds of papers have studied the limit cycles of planar polynomial differential systems. The main reason of these studies is the unsolved 16^{th} Hilbert problem, see [7,8,10]. In this paper, we will try to give a partial answer to this problem for the class of generalized Liénard polynomial differential system given by

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$$\begin{cases} \dot{x} = y + h(x), \\ \dot{y} = -x - f(x)y^{2p+1} - g(x), \end{cases} \tag{1}$$

where $h(x), f(x)$ and $g(x)$ are polynomials in the variable x of degree l, n and m respectively and $p \in \mathbb{N}$. This system was studied when $h(x) = 0$ in [1]. [12] and [13] considered the similar case of differential system (1) for $p = 0$.

Note that when $h(x) = g(x) = 0$ and $p = 0$ system (1) coincides with the classical polynomial Liénard differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - f(x)y, \end{cases} \tag{2}$$

where $f(x)$ is a polynomial in the variable x of degree n. A generalization of classical Liénard differential system (2) is the following system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -f(x)y - g(x), \end{cases} \tag{3}$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degree n and m respectively. We denote by $H(m, n)$ the maximum number of limit cycles of system (3). This number is usually called the Hilbert number for system (3).

- In 1928, Liénard [14] proved that if $m = 1$ and $F(x) = \int_0^x f(s)\mathrm{d}s$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then system (3) has a unique limit cycle.
- In 1973, Rychkov [19] proved that if $m = 1$ and $f(x)$ is an odd polynomial of degree five, then system (3) has at most two limit cycles.
- In 1977, Lins de Melo and Pugh [15] proved that $H(1, 1) = 0$ and $H(1, 2) = 1$.
- In 1988, Coppel [5] proved that $H(2, 1) = 1$.
- In 1997, Dumortier and Li [6] proved that $H(3, 1) = 1$.
- In 2012, Li and Llibre [11] proved that $H(1, 3) = 1$.

A well known method for obtaining results on the limit cycles of polynomial differential systems perturbs the linear center $\dot{x} = y,\ \dot{y} = -x$ inside the class of polynomial differential systems, or inside the class of classical polynomial Liénard differential systems. The limit cycles obtained in this way are sometimes called medium amplitude limit cycles.