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# Decay Rate Toward the Traveling Wave for Scalar Viscous Conservation Law

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**Abstract.** The time-decay rate toward the viscous shock wave for scalar viscous conservation law

$$u_t+f(u)_x=\mu u_{xx}$$

is obtained in this paper through an $L^p$ estimate and the area inequality in [1] provided that the initial perturbations are small, i.e., $\|\Phi_0\|_{H^2}\leq\varepsilon$, where $\Phi_0$ is the anti-derivative of the initial perturbation. It is noted that there is no additional weighted requirement on $\Phi_0$, i.e., $\Phi_0(x)$ only belongs to $H^2(R)$.

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**Key words**: Viscous conservation law, shock wave, decay rate.

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## 1 Introduction

In this paper, we are concerned with the Cauchy problem of the viscous conservation law, which reads as,

$$\begin{cases} u_t+f(u)_x=\mu u_{xx}, \\ u(0,x)=u_0(x)\to u_\pm \quad \text{as} \quad x\to\pm\infty, \end{cases} \tag{1.1}$$

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where $u(x,t)$ is the unknown function, $f(u)$ is the so-called flux function, which is a given smooth function, $u_0(x)$ is the initial data, $\mu\in\mathbb{R}^+$ is the viscosity coefficient and $u_\pm$ are the far field states.

It is well-known that the long time behavior of solutions for the Cauchy problem (1.1) is closely related to the corresponding Riemann solutions, denoted as $u^R(x)$, of the Riemann problem

$$\begin{cases} u_t+f(u)_x=0, \\ u(0,x)=u_0^R(x), \end{cases} \tag{1.2}$$

where $u_0^R(x)$ is the Riemann initial data given by

$$u_0^R(x)=\begin{cases} u_-, & x<0, \\ u_+, & x>0. \end{cases} \tag{1.3}$$

The Riemann solutions contain two kinds of basic wave patterns, i.e., shock and rarefaction waves. In this paper, we focus on the shock wave case. Due to the effect of viscosity in (1.1), the shock wave is smoothed as a smooth function, called by viscous shock wave (or traveling wave). The stability of viscous shock wave has been extensively studied and many important achievements were obtained. Indeed, Il'in-Oleinik proved in 1960's that if $f(u)$ is strictly convex, i.e., $f''(u)>0$, the solution of (1.1) time-asymptotically tends to the viscous shock wave. Kawashima-Matsumura [8] further obtained the convergence rate if the initial data belongs to a weighted Sobolev space, see also [15] for the case that $f(u)$ is not convex or concave. An interesting $L^1$ stability theorem was established in [3].

Considerable progress on the asymptotic stability of traveling waves has been further achieved for the systems of viscous conservation laws such as compressible Navier-Stokes system since the pioneer works of Goodman [2] and Matsumura-Nishihara [14], see [4,5,8,10–12,18–20] and the references therein. In particular, Liu-Zeng [12] obtained the pointwise estimates in the stability analysis of viscous shock wave through approximate Green function approach and pointwise estimates.

Nevertheless, it is also interesting to study the decay rates toward the viscous shock wave through the basic energy method. To the best of our knowledge, Kawashima-Matsumura [8] first obtained the decay rate for scalar viscous conservation law (1.1) by a weighted energy method. Since then, there have been several works on the decay properties toward the viscous shock, cf. [7,15,16], in which all of the decay rates in time depend on the decay rates of the initial data