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Regularity Criteria of the Solutions to Axisymmetric Magnetohydrodynamic System

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Abstract In this paper, we consider Cauchy problem of the axially symmetric Magnetohydrodynamic (MHD) system. By using energy method, we establish some regularity criteria of the solutions for the axisymmetric solutions of the three dimensional incompressible MHD system.

Keywords Axisymmetric solutions, Regularity criterion, Incompressible magnetohydrodynamics.

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1. Introduction

In this paper, we consider Cauchy problem of the incompressible MHD system:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \triangle u + \nabla p = (v\cdot\nabla)B, \\ \partial_t B + (u\cdot\nabla)B - \triangle B = (B\cdot\nabla)u, \\ \text{div}u = 0,\ \text{div}B = 0, \\ (u,B)(\mathbf{x},0) = (u_0, B_0), \end{cases} \tag{1.1}$$

where $u = u(\mathbf{x},t) \in \mathbb{R}^3$ denotes the velocity of the fluid, $B = B(\mathbf{x},t) \in \mathbb{R}^3$ stands for the magnetic field, the scalar function $p = p(\mathbf{x},t)$ is pressure and $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$.

Our main concern here is to establish a family of unique solutions of system (1.1) with the form

$$u(\mathbf{x},t) = u_r(r,z,t)e_r + u_\theta(r,z,t)e_\theta + u_z(r,z,t)e_z,$$

$$B(\mathbf{x},t) = B_r(r,z,t)e_r + B_\theta(r,z,t)e_\theta + B_z(r,z,t)e_z,$$

in the cylindrical coordinate system. Here,

$$e_r = (\frac{x}{r}, \frac{y}{r}, 0),\ e_\theta = (-\frac{y}{r}, \frac{x}{r}, 0),\ e_z = (0,0,1),$$
$$r = \sqrt{x^2+y^2},\ (x,y,z) = (r\cos\theta, r\sin\theta, z).$$

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In terms of $(u_r, u_\theta, u_z, B_r, B_\theta, B_z)$, the MHD system (1.1) can be rewritten as

$$\begin{cases} \partial_t u_r + (u\cdot\nabla)u_r - \triangle u_r + \frac{u_r}{r^2} - \frac{u_\theta^2}{r} + \partial_r p - (B\cdot\nabla)B_r + \frac{B_\theta^2}{r} = 0, \\ \partial_t u_\theta + (u\cdot\nabla)u_\theta - \triangle u_\theta + \frac{u_\theta}{r^2} + \frac{u_r u_\theta}{r} - (B\cdot\nabla)B_\theta - \frac{B_r B_\theta}{r} = 0, \\ \partial_t u_z + (u\cdot\nabla)u_z - \triangle u_z + \partial_z p - (B\cdot\nabla)B_z = 0, \\ \partial_t B_r + (u\cdot\nabla)B_r - \triangle B_r + \frac{B_r}{r^2} - (B\cdot\nabla)u_r = 0, \\ \partial_t B_\theta + (u\cdot\nabla)B_\theta - \triangle B_\theta + \frac{B_\theta}{r^2} + \frac{u_\theta B_r}{r} - (B\cdot\nabla)u_\theta - \frac{B_\theta u_r}{r} = 0, \\ \partial_t B_z + (u\cdot\nabla)B_z - \triangle B_z - (B\cdot\nabla)u_z = 0, \\ \frac{u_r}{r} + \partial_r u_r + \partial_z u_z = 0, \\ \frac{B_r}{r} + \partial_r B_r + \partial_z B_z = 0, \\ (u_r, u_\theta, u_z, B_r, B_\theta, B_z)(r,z,0) = (u_0^r, u_0^\theta, u_0^z, B_0^r, B_0^\theta, B_0^z)(r,z). \end{cases} \tag{1.2}$$

First, let us briefly review some results of the MHD system (1.1). For the 2D case, Jiu and Zhao [9] get a global regular solution of MHD with dissipation terms $-(-\triangle)^\alpha u$ and $-(-\triangle)^\beta B$, when $0 \le \alpha < \frac{1}{2}, \beta \ge 1, 3\alpha + 2\beta > 3$. In particular, they also prove the solution exists globally when $\alpha = 0$ and $\beta > \frac{3}{2}$. For more details, one may refer to [1,6,15,19,20,24]. In the 3D case, Zhou [23] gets the MHD system with dissipation terms $-(-\triangle)^\alpha u$ and $-(-\triangle)^\beta B$ satisfying the following conditions:

$$u(x,t) \in L^p(0,T;L^q(\mathbb{R}^3)),\ \frac{2\alpha}{p} + \frac{3}{q} \le 2\alpha - 1,\ \frac{3}{2\alpha - 1} < q \le \infty,$$

or

$$\Lambda^\alpha u(x,t) \in L^p(0,T;L^q),\ \frac{2\alpha}{p} + \frac{3}{q} \le 3\alpha - 1,\ \frac{3}{3\alpha - 1} < q \le \frac{3}{\alpha - 1}.$$

Then, the solution is globally regular. Other regularity criteria were shown in [4,7,8,21,22,25].

When $B = 0$, system (1.1) reduce to axisymmetric Navier-Stokes system, there have been extensive studies on the regularity criteria for axisymmetric Navier-Stokes, cf. [2, 5, 10–12, 16, 18]. Here, we only mention some results that related to our main results. Firstly, Chae and Lee [2] obtain some regularity criteria for axisymmetric weak solutions of the 3D Navier-Stokes equations with nonzero swirl. In [18], Wei proves the global regularity of solutions to the axially symmetric Navier-Stokes equations, if $\|ru_\theta(r,z,0)\|_{L^\infty}$ or $\|ru_\theta(r,z,t)\|_{L^\infty(r\le r_0)}$ is smaller than some dimensionless quantity of the initial data. This result improves the one in Lei and Zhang [12]. Motivated by [18], we will study a global regularity for the axially symmetric MHD system.

For equation (1.2), Lei [10] proves the global well-posedness of classical solutions for a family of special axisymmetric initial data whose swirl components of the velocity field and magnetic vorticity field are trivial. In [16], Wang and Wu study the properties of solutions to axially symmetric incompressible MHD system in three dimensions, and construct a family of global smooth solutions by applying the one-dimensional solutions. Recently, Li and Yuan [13] have obtained regularity criteria for the axisymmetric solutions of MHD system, if $\omega_\theta \in L^q(0,T;L^p(\mathbb{R}^3))$, and $n_\theta \in L^q(0,T;L^p(\mathbb{R}^3))$ satisfy

$$\int_0^T (\|\omega_\theta\|_p^q + \|n_\theta\|_p^q)dt < \infty,\ with\ \frac{3}{p} + \frac{2}{q} \le 2,\ \frac{3}{2} < p \le \infty,\ 0 < q < \infty.$$