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Solvability and Stability for Singular Fractional (p, q)-difference Equation*

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Abstract In this paper, we initiate the solvability and stability for a class of singular fractional (p, q)-difference equations. First, we obtain an existence theorem of solution for the fractional (p, q)-difference equation. Then, by using a fractional (p, q)-Gronwall inequality, some stability criteria of solution are established, which also implies the uniqueness of solution.

Keywords Fractional (p, q)-difference equation, Existence of solution, Stability, (p, q)-Gronwall inequality, Uniqueness.

MSC(2010) 34A12, 39A30, 39A70.

1. Introduction

The q-difference is a kind of discrete calculus. In the early 20th century, the q-difference was first systematically studied by Jackson [9]. After the 1970s, the theory of q-difference has been extensively studied. Mitlagel Leffler proposed the theory of fractional q-difference operators, and later the related theories such as q-Laplace and Fourier Transform, q-Sturm-Liouville theory, q-Taylor expansion, q-Bernstein polynomial and so on attracted a great deal of attention [2,7,10]. In recent years, q-difference has been more and more frequently used in natural science and engineering. It plays an important role in mathematical physical models, dynamical systems, quantum physics and economics. For more details, the reader may refer to [8,12,15].

Motivated by these applications of q-calculus which is also called quantum calculus, many researchers have developed the theory of quantum calculus based on two-parameter p and q. In 1991, Chakrabarti and Jagannathan first investigated the (p, q)-calculus in quantum algebras [5]. For some results on the study of (p, q)-calculus, we refer to [11, 13, 14, 17]. The (p, q)-calculus is used efficiently in many fields such as physical sciences, hypergeometric series, lie group, special functions, approximation theory, Bezier curves and surfaces and etc.

The problem of fractional calculus in discrete settings has become an active research area [1,3,6]. Agarwal [1] and Al-Salam [3] introduced fractional q-difference calculus, while Diaz and Osler [6] studied fractional difference calculus. Recently, Brikshavana and Sitthiwirattham havd studied the fractional Hahn calculus [4]. In

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2020, Soontharanonl and Sitthiwirattham introduced the fractional (p, q)-calculus [18]. In the meantime, they studied the existence of a fractional (p, q)-difference equation. Compared with q-difference equations, (p, q)-difference equations have two quantization parameters p and q, which are genuinely independent. They have wider applicability in concrete mathematical models of quantum mechanics and fluid mechanics [5].

Although many interesting results related to discrete analogues of some topics of continuous fractional calculus have been studied, the theory of discrete fractional calculus remains much less developed than that of continuous fractional calculus. In particular, there are few papers about fractional (p, q)-calculus. As far as we know, the stability of fractional (p, q)-difference equation has not been studied, even for regular fractional (p, q)-difference equation. Up to now, no research has existed about solvability for Caputo type fractional (p, q)-difference equations. The gap mentioned is the motivation for this research.

In this paper, we consider the solvability and stability of the fractional (p, q)-difference equation:

$$\begin{cases} {}^cD^{\alpha}_{p,q}x(t) = f\left(t, x(t)\right), \ t > 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $0 < q < p \leq 1$, and ${}^cD^{\alpha}_{p,q}$ is Caputo type fractional (p, q)-difference operator. In this paper, we first prove that the fractional (p, q)-difference equation has at least one solution if $t^{\alpha}f(t, x)$ is continuous on variables t and x by Ascoli-Arzela's lemma. Furthermore, we establish a fractional (p, q)-Gronwall inequality. By the fractional (p, q)-Gronwall inequality, we obtain a stability criterion.

This paper is structured as follows: In Section 2, we present necessary definitions, properties and lemmas. In Section 3 and Section 4, some results on the existence of solution and stability are obtained. An example is given in Section 5. Finally, we end the paper with a conclusion.

2. Preliminaries

In this section, we present basic definitions, notations, and lemmas that will be used in this paper. Let $0 < q < p \leq 1$. We introduce the notation [18]:

$$[k]_{p,q} := \begin{cases} \dfrac{p^k - q^k}{p - q} = p^{k-1}[k]_{\frac{q}{p}}, & k \in \mathbb{N}, \\ 1, & k = 0, \end{cases}$$

and the (p, q)-analogue factorial is defined as:

$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q} \cdots [1]_{p,q} = \displaystyle\prod_{i=1}^{k} \frac{p^i - q^i}{p - q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

The (p, q)-analogue of the power function $(a - b)^{(n)}_{p,q}$ with $n \in N_0 := \{0, 1, 2, \ldots\}$ is given by

$$(a - b)^{(0)}_{p,q} := 1, \qquad (a - b)^{(n)}_{p,q} := \prod_{k=0}^{n-1}(ap^k - bq^k), \qquad a, b \in \mathbb{R}.$$