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Existence of Solutions to a Class of Fractional Differential Equations*

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Abstract In this paper, the existence of solutions to a class of fractional differential equations $D^{\alpha}_{0+}u(t) = h(t)f(t, u(t), D^{\theta}_{0+}u(t))$ is obtained by an efficient and simple monotone iteration method. At first, the existence of a solution to the problem above is guaranteed by finding a bounded domain D_M on functions f and g. Then, sufficient conditions for the existence of monotone solution to the problem are established by applying monotone iteration method. Moreover, two efficient iterative schemes are proposed, and the convergence of the iterative process is proved by using the monotonicity assumption on f and g. In particular, a new algorithm which combines Gauss-Kronrod quadrature method with cubic spline interpolation method is adopted to achieve the monotone iteration method in Matlab environment, and the high-precision approximate solution is obtained. Finally, the main results of the paper are illustrated by some numerical simulations, and the approximate solutions graphs are provided by using the iterative method.

Keywords Fractional differential equation, Monotone iteration method, Numerical simulation, Approximate solutions graphs.

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1. Introduction

At present, fractional differential equation (FDE) is a focused issue concerned by many researchers around the world owing to their various applications in physics, chemistry, biology, dynamical control, engineering and medicine, etc. [1, 10, 19, 22, 24, 28]. Although considerable attention has been paid to the solutions of FDEs (see [3, 6, 9, 14, 16–18, 21, 25, 27] and the references), there are few works [3, 9, 18] on numerical methods which are used to compute approximate solutions of the FDEs whose nonlinear term involves the derivative.

For integral order ordinary differential equations (ODEs), there are many efficient numerical methods, such as Euler method, extrapolation method, monotone iteration method, variational iteration method [2, 4, 5, 7, 12, 13, 20, 23, 26], but it

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is very difficult to solve or to compute approximate solutions of the ODEs whose nonlinear term involves the derivative. For FDEs, it is even harder to obtain approximate solutions. Therefore, the study on numerical methods for FDEs whose nonlinear term involves the derivative is of theoretical and practical significance.

In the first place, literature review has been made on some related studies. In [23], Yao studied the following problem

$$\begin{cases} u^{(4)} = f(t, u(t), u'(t)), \quad t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

where $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous. With the help of the improved monotonically iterative method, Yao dealt with the nonlinear boundary value problem and obtained the existence and iteration of positive solution to the nonlinear problem. He also provided a useful computational method.

In [7], Edson et al., investigated the following two fourth-order problems. The first problem is

$$\begin{cases} u^{(4)} = f\left(t, u(t), u'(t)\right), \quad t \in (0,1)\,, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = g(u(1)). \end{cases}$$

The second one is

$$\begin{cases} u^{(4)} = f\left(t, u(t), u'(t)\right), \quad t \in (0,1)\,, \\ u(0) = u'(0) = 0, \\ u'(1) = 0, \quad u'''(1) = g(u(1)). \end{cases}$$

By using monotone iteration method, the authors proposed a numerical method to compute approximate solutions and obtained monotone positive solutions.

In [26], Zhang discussed an elastic beam equation with a corner

$$\begin{cases} u^{(4)} = q(t)f(t, u(t), u'(t)), \quad t \in (0,1), \\ u(0) = u'(0) = u'(1) = u'''(1) = 0, \end{cases}$$

where $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, and $q(t) : (0,1) \to [0,\infty)$ is a continuous function satisfying $0 < \int_0^1 s^2 q(s)ds < \infty$. By applying monotone iterative techniques, Zhang constructed a successive iterative scheme whose starting point is a simple quadratic function or a zero function, and obtained the existence and iteration of positive solutions to the above boundary value problem.

Motivated by the ideas mentioned above, in this paper, we consider the following FDE whose nonlinear term involves the derivative

$$D^{\alpha}_{0+}u(t) = h(t)f(t, u(t), D^{\theta}_{0+}u(t)),\ t \in [0,1] \tag{1.1}$$

with either the boundary value conditions or the initial value conditions

$$\mathit{\Gamma}_1(t, u(t), D^{\beta}_{0+}u(t)) = 0, \cdots, \mathit{\Gamma}_n(t, u(t), D^{\beta}_{0+}u(t)) = 0, \tag{1.2}$$