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Asymptotics of a Class of Singularly Perturbed Weak Nonlinear Boundary Value Problem with a Multiple Root of the Degenerate Equation*

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Abstract A singularly perturbed boundary value problem with weak nonlinearity in the case when the degenerate equation has a multiple root is studied. The asymptotic approximation of the solution is constructed by the modified boundary layer function method. Based on the comparison principle, there exist multizonal boundary layers in the neighborhood of the endpoints. The existence of a solution is proved by using the method of asymptotic differential inequalities.

Keywords Singularly perturbed problem, Multiple root of the degenerate equation, Asymptotic method, Upper and lower solutions.

MSC(2010) 34B15, 34b16, 34E05, 34E15.

1. Problem statement

It is well-known that many scholars [9, 10] have been attracted to the study of ordinary differential equations. With the development of the times, it has been found out that there are many mathematical models with small parameters in practical problems. In particular, the singularly perturbed reaction-advection-diffusion equation plays an important role in practical application such as the propagation, decay and chemical reaction of impurities in the atmosphere [8]. Therefore, this kind of problem has attracted the attention of a great many of mathematical experts and scholars. To the best of our knowledge, a lot of research in the case when the degenerate equation has isolated roots has been carried out [7, 12, 14–18]. When the degenerate equation has multiple roots, the critical manifold is not normally hyperbolic, and cannot meet the stability condition of Tikhonov's theorem. In this case, it is necessary to use a modified boundary layer method to resolve difficulties. As shown in [1–6, 19], the boundary layers can be decomposed into three zones, and their formal asymptotic solutions have different decay characters with respect to diverse scales in distinct regions. The singularly perturbed reaction-diffusion equa-

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tions without advection terms in the case of multiple roots have been studied in [2,3]. In this paper, the stationary problem for a class of reaction-advection-diffusion equation with weak nonlinearity and multiple root of the degenerate equation is considered:

$$\begin{cases} \varepsilon^2 \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} - a(x)\left(\varepsilon \dfrac{\mathrm{d}u}{\mathrm{d}x}\right)^2 = f(u, x, \varepsilon), \quad 0 < x < 1, \\ u(0, \varepsilon) = u^0, \quad u(1, \varepsilon) = u^1, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, and u is a scalar function.

This singularly perturbed problem is sophisticated, and the modified boundary layer function method shall be applied to construct the asymptotic solution. More importantly, we obtain a result that the nature of boundary layer with a transition from algebraic decay to exponential decay by comparison principle. Finally, the existence of a solution is proved by the method of upper and lower solutions [11,13].

Let the following assumptions be satisfied.

Denote

$$\bar{D} = \{(u, x, \varepsilon) |\ |u| \le l,\ 0 \le x \le 1,\ 0 \le \varepsilon \le \varepsilon_0\}.$$

Assumption 1.1. *Let*

$$f(u, x, \varepsilon) = h(u, x)(u - \varphi(x))^2 - \varepsilon f_1(u, x, \varepsilon), \tag{1.2}$$

where the functions $h(u, x)$, $\varphi(x)$ and $f_1(u, x, \varepsilon)$ are sufficiently smooth on the set $\bar{D}$, and one has the inequality

$$\bar{f}_1(x) := f_1(\varphi(x), x, 0) > 0, \quad 0 \le x \le 1.$$

Moreover, $h(u, x)$ conforms to one of the following requirements:

(i)

$$h(u, x) > 0, \quad \varphi(x) \le u < \psi(x), \quad 0 \le x \le 1,$$

where $\psi(x)$, $0 \le x \le 1$ is a function that satisfies $\psi(x) > \phi(x)$, $h(\psi(x), x) = 0$;

(ii)

$$h(u, x) > 0, \quad u \ge \varphi(x), \quad 0 \le x \le 1,$$

.

To be definite, we shall consider case (ii).

Assumption 1.1 shows that the degenerate equation

$$f(u, x, 0) = 0 \tag{1.3}$$

has a multiple root

$$\bar{u}(x) = \varphi(x), \quad 0 \le x \le 1. \tag{1.4}$$

To determine the leading term in the asymptotic representation of the boundary layers in the course of constructing the asymptotics of solution to problem (1.1), the following condition is needed.

Assumption 1.2. *Assume that the inequalities are satisfied:*

$$u^0 - \varphi(0) > 0, \quad u^1 - \varphi(1) > 0;$$

$$a(x) < 0, \quad 0 \le x \le 1.$$