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Dynamical Behavior and Exact Traveling Wave Solutions for Three Special Variants of the Generalized Tzitzeica Equation*

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Abstract The dynamics and bifurcations of traveling wave solutions are studied for three nonlinear wave equations. A new phenomenon, such as a composed orbit, which consists of two or three heteroclinic orbits, may correspond to a solitary wave solution, a periodic wave solution or a peakon solution, is found for the equations. Some previous results are extended.

Keywords Generalized Tzitzeica equation, Solitary wave solution, Periodic wave solution, Peakon solution.

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1. Introduction

The generalized Tzitzeica equation

$$u_{xt} = ae^{mu} + be^{nu}, \tag{1.1}$$

where $a,\ b$ are arbitrary constants and $m,\ n$ are integers. Specially, when $m = 1$ and $n = -2$, equation (1.1) is called the Tzitzeicta equation, which was originally found from the work of Tzitzeica [14]. Later on, Dodd, Bullough and Mikhailov [4, 12] introduced some other forms of the evolution equation involving the exponential term e^{pu}. For $m = 1$ and $n = -1$, equation (1.1) reduces to the sinh-Gordon equation. For $m = 1$ and $n = -2$, equation (1.1) turns into the Dodd–Bullough–Mikhailov equation. For $m = -1$ and $n = -2$, equation (1.1) gives rise to the Tzitzeica–Dodd–Bullough equation. These well-known equations have extensive applications in mathematical biology, nonlinear optics, fluid mechanics, chemical kinetics and quantum field theory [7, 13, 18]. Particularly in recent years, many researchers have focused on the search for exact solutions of nonlinear wave equations by using different methods and techniques. Among others, Wazwaz [15] derived some traveling wave solutions for the above three variants of the generalized Tzitzeica equation (1.1) by employing the tanh method and extended tanh method.

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Abazari [1] applied the (G'/G)-method for constructing solitons and periodic solutions for the generalized Tzitzeica equation (1.1). In fact, it is important to understand the dynamical behavior of traveling wave solutions for nonlinear wave equations [3, 5, 6, 8–11, 16, 17, 19–22]. In [5], Chen et al., studied the qualitative behavior of the traveling wave solutions of the generalized Tzitzeica equation (1.1) in the case of degenerate equilibrium points.

In this paper, we carry out further study on the dynamical behavior and exact traveling wave solutions of the three variants of equation (1.1). By presenting some representative bifurcation diagrams under different parametric conditions, not only some new exact explicit expressions of traveling wave solutions including solitary wave solutions, periodic wave solutions and peakon solutions are obtained, but also some interesting phenomena arise. For instance, a composed orbit, which consists of two or three heteroclinic orbits, may correspond to a solitary wave solution, a periodic wave solution or a peakon solution. This work complements the previous results of [5].

2. Phase portraits of three variants of the generalized Tzitzeica equation

To study the equilibrium points and their properties of the corresponding traveling wave system of the generalized Tzitzeica equation, we need to introduce some preliminaries [2] first.

Lemma 1. Suppose $p \in R$ be an equilibrium point of the planar polynomial integrable system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y). \tag{2.1}$$

Denote $\Delta = P_x(p)Q_y(p) - P_y(p)Q_x(p)$ and $T = P_x(p) + Q_y(p)$. If $\Delta < 0$, then p is a saddle point. If $T^2 > 4\Delta > 0$, then p is a node (stable if $T < 0$, unstable if $T > 0$). If $T = 0 < \Delta$, then p is a center. Moreover, if $\Delta = T = 0$ and the Jacobian matrix at the point p is not the zero matrix, then p is a nilpotent point.

2.1. Phase portraits of the sinh-Gordon equation

For $m = 1$ and $n = -1$, equation (1.1) reduces to the sinh-Gordon equation

$$u_{xt} = ae^u + be^{-u}. \tag{2.2}$$

We look for the traveling wave solutions of equation (2.2) in the form of $u(x, t) = u(\xi)$ with $\xi = x - ct$. Inserting it into equation (2.2) and subsequently making the variable transformation $v = e^u$, we have

$$cvv'' - c(v')^2 + av^3 + bv = 0. \tag{2.3}$$

Letting $y = v'$ in equation (2.3) generates a planar system

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{cy^2 - av^3 - bv}{cv} \tag{2.4}$$

with the first integral

$$H(v, y) = \frac{cy^2 + 2av^3 - 2bv}{cv^2} = h. \tag{2.5}$$