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The Regularity of Stochastic Convolution Driven by Tempered Fractional Brownian Motion and Its Application to Mean-field Stochastic Differential Equations*

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Abstract In this paper, some properties of a stochastic convolution driven by tempered fractional Brownian motion are obtained. Based on this result, we get the existence and uniqueness of stochastic mean-field equation driven by tempered fractional Brownian motion. Furthermore, combining with the Banach fixed point theorem and the properties of Mittag-Leffler functions, we study the existence and uniqueness of mild solution for a kind of time fractional mean-field stochastic differential equation driven by tempered fractional Brownian motion.

Keywords Mean-field stochastic differential equations, Tempered fractional Brownian motion, Caputo fractional derivative, Banach fixed point theorem.

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1. Introduction

In 2015, Sabzikar, Meerschaert and Chen [16] proposed a tempered fractional derivative that multiplies the power law kernel by an exponential tempering factor. Temper means that it produces a more tractable mathematical object, and can be made arbitrarily light. It has better properties than the fractional derivative that can be approximated by adjusting the parameters by resulting operator over a finite interval. The basic definitions and properties of tempered fractional Brownian motion (TFBM for its abbreviated form) was proposed in [12], which modifies the power law kernel in the moving average representation of a fractional Brownian motion adding an exponential tempering. Then, the theories of stochastic integrals for TFBM were developed in [13]. Along the way, they also developed some basic results on tempered fractional calculus. First, we give the definition of TFBM, as defined in [12]. Detailed description will be introduced in Section 2.

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Definition 1.1. For any $\sigma < 1/2$ and $\lambda > 0$, letting $\{B(t)\}_{t\in\mathbb{R}}$ be a real-valued Brownian motion on the real line, the stochastic integral

$$B^{\sigma,\lambda}(t) = \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-x)_+}(t-x)_+^{-\sigma} - e^{-\lambda(-x)_+}(-x)_+^{-\sigma}\right] B(dx)$$

is called a TFBM, where $(x)_+ = xI_{(x>0)}, 0^0 = 0$, and λ is called tempered parameter.

A solution to stochastic evolution equations can usually be as represented a stochastic convolution generated by infinitesimal generator driven by noises. It is requisite to give the regularity of stochastic convolution in function spaces where the solution exists. In this paper, we are concerned about the regularity of stochastic convolution driven by TFBM (Lemma 3.2 and Lemma 3.3). As an application, the existence and uniqueness of the following mean-field stochastic differential equations (SDEs for its abbreviated form) driven by TFBM is studied.

$$\begin{cases} dx(t) = Ax(t)dt + f\left(t, x(t), \mathbb{P}_{x(t)}\right)dt + \gamma\left(t, \mathbb{P}_{x(t)}\right)dB_Q^{\sigma,\lambda}(t), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where the operator A generates an exponentially stable semigroup, f, γ are continuous functions with additional properties which would be specified below. Assume that there exists a complete orthogonal basis $\{e_k\}_{k\in\mathbb{N}}$ in a Hilbert space $\mathbb{H}$, and that $B_Q^{\sigma,\lambda} = \left\{B_Q^{\sigma,\lambda}(t)\right\}_{t\geq 0}$ is cylindrical $\mathbb{H}$ -valued TFBM defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ with a finite trace nuclear covariance operator $Q \geq 0$. Denote $\mathrm{Tr}(Q) = \Sigma_{k=1}^{\infty}\lambda_k < \infty$, which satisfies that $Qe_k = \lambda_k e_k, k \in \mathbb{N}$. The cylindrical $\mathbb{H}$ -valued TFBM is defined as

$$B_Q^{\sigma,\lambda}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} B_k^{\sigma,\lambda}(t) e_k, \quad t \geq 0,$$

where $\{B_k^{\sigma,\lambda}\}$ $(k \in \mathbb{N})$ are independent TFBMs.

Mean-field SDEs, also named as McKean-Vlasov equations, whose coefficients depend upon the marginal distribution of the solution, were discussed the first time by Kac, Uhlenbeck and Hibb [7] in their analysis of the Boltzmann equation for the particle density in dilute monotonic gases and a toy model for the Vlasov kinetic equation of plasma. Mean field SDEs have been used to study high-dimensional systems corresponding to a large number of particles, i.e., N-players stochastic differential games [9, 10]. The existence and uniqueness of almost automorphic solutions in distribution of mean field SDEs driven by fractional Brownian motion were established in [2]. The existence of stationary solutions adapted to dissipative finite-dimensional SDEs driven by TFBM was studied in [5, 6]. Recently, Wang, Liu and Caraballo [19] have considered the exponential behavior and upper noise excitation index of solutions to evolution equations with unbounded delay and tempered fractional Brownian motions. As far as we are concerned, it is meaningful to discuss the global existence and uniqueness of mean-field stochastic differential equation driven by TFBM. In addition, it has been found that fractional calculus can be useful in the most diverse areas of science, mainly due to the nonlocal character of the fractional differentiation. There have been many meaningful results in recent years. Carvalho and Planas [1] got the mild solutions to the time fractional Navier-Stokes equations in $\mathbb{R}^n$. Shen and Huang [18] discussed time-space fractional