Analysis in Theory and Applications
Anal. Theory Appl., Vol. **38**, No. 3 (2022), pp. 335-350

DOI: 10.4208/ata.OA-2017-0080
September 2022

New Results on Generalized c-Distance without Continuity in Cone b-Metric Spaces over Banach Algebras

Yan Han[1] and Shaoyuan Xu[2,*]

[1] *School of Mathematics and Statistics, Zhaotong University, Zhaotong, Yunnan 657000, China*
[2] *School of Mathematics and Statistics, Hanshan Normal University, Chaozhou, Guangdong 521041, China*

Received 4 December 2017; Accepted (in revised version) 24 September 2019

Abstract. In this work, some new fixed point results for generalized Lipschitz mappings on generalized c-distance in cone b-metric spaces over Banach algebras are obtained, not acquiring the condition that the underlying cone should be normal or the mappings should be continuous. Furthermore, the existence and the uniqueness of the fixed point are proven for such mappings. These results greatly improve and generalize several well-known comparable results in the literature. Moreover, some examples and an application are given to support our new results.

Key Words: Cone b-metric spaces over Banach algebras, generalized c-distance, non-normal cone, generalized Lipschitz mappings, fixed point theorems.

AMS Subject Classifications: 54H25, 47H10

1 Introduction and preliminaries

In 2007, Huang and Zhang [1] introduced the notion of cone metric space as a generalization of metric space. Subsequently, [7] expanded the work of [1] into cone b-metric space with coefficient $s \geq 1$ which greatly generalized cone metric space. Later on, many authors established various fixed point theorems in cone metric (or cone b-metric) spaces (see [4,17] and references mentioned therein). Since 2010, some scholars showed that the fixed point results in cone metric spaces with the assumption of normal cone are the simple consequences of corresponding results of usual metric spaces (see [2,10]). It is not a hot topic. Fortunately, Liu and Xu [13] firstly defined cone metric space over Banach algebra and proved some fixed point theorems in such spaces. Moreover, they also gave an

*Corresponding author. *Email addresses:* `hanyan702@126.com` (Y. Han), `xushaoyuan@126.com` (S. Xu)

 ©2022 Global-Science Press

example which showed that fixed point theorems in cone metric spaces over Banach algebras are not equivalent to those in (general) metric spaces. In 2011, Cho et al. [5] and Wang and Guo [19] introduced the concept of c-distance which is a cone version of w-distance of Kada et al. [6]. Then, lots of fixed point results on c-distance in cone metric spaces, cone b-metric spaces and tvs-cone metric spaces were introduced in [8,9,12,15,18]. Very recently, Bao et al. [4] defined generalized c-distance in cone b-metric spaces and obtained several fixed point results in ordered cone b-metric spaces. However, the conditions of these theorems relied strongly on the assumptions that the underlying cone is normal or the mappings are continuous.

In this work, we obtain the existence of the fixed point but also the uniqueness on generalized c-distance in cone b-metric spaces over Banach algebras, without the assumption of normality of the cone and the notion of continuity of the mappings at the same time. The main results improve and generalize some important known results in the literature [5,8–10,12,15–19]. Moreover, some examples and an application are given to show that the main results are indeed real improvements and generalizations of the corresponding results in the literature.

First, we recall some basic terms and definitions about Banach algebras and cone metric spaces.

Let $\mathcal{A}$ be a real Banach algebra, i.e., $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties: for all $x, y, z \in \mathcal{A}$, $a \in \mathbb{R}$

(1) $x(yz) = (xy)z$;

(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(3) $a(xy) = (ax)y = x(ay)$;

(4) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that the Banach algebra $\mathcal{A}$ has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [20].

The following proposition is well known (see [20]).

Proposition 1.1. *Let $\mathcal{A}$ be a real Banach algebra with a unit e and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of $\mathcal{A}$ is called a cone if :