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A NURBS-Enhanced Finite Volume Method for Steady Euler Equations with Goal-Oriented h-Adaptivity

Xucheng Meng[1,2] and Guanghui Hu[3,4,5,*]

[1] *Research Center for Mathematics, Beijing Normal University, Zhuhai 519087, China.*
[2] *BNU-HKBU United International College, Zhuhai 519087, China.*
[3] *Department of Mathematics, University of Macau, Macao SAR, China.*
[4] *Zhuhai UM Science and Technology Research Institute, Zhuhai, Guangdong Province, China.*
[5] *Guangdong-Hong Kong-Macao Joint Laboratory for Data-Driven Fluid Dynamics and Engineering Applications, University of Macau, Macao SAR, China.*

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Abstract. In [*A NURBS-enhanced finite volume solver for steady Euler equations, X. C. Meng, G. H. Hu, J. Comput. Phys., Vol. 359, pp. 77–92*], a NURBS-enhanced finite volume method was developed to solve the steady Euler equations, in which the desired high order numerical accuracy was obtained for the equations imposed in the domain with a curved boundary. In this paper, the method is significantly improved in the following ways: (i) a simple and efficient point inversion technique is designed to compute the parameter values of points lying on a NURBS curve, (ii) with this new point inversion technique, the h-adaptive NURBS-enhanced finite volume method is introduced for the steady Euler equations in a complex domain, and (iii) a goal-oriented *a posteriori* error indicator is designed to further improve the efficiency of the algorithm towards accurately calculating a given quantity of interest. Numerical results obtained from a variety of numerical experiments with different flow configurations successfully show the effectiveness and robustness of the proposed method.

AMS subject classifications: 76M12, 65N50, 35Q31, 65D07

Key words: Steady Euler equations, NURBS-enhanced finite volume method, goal-oriented *a posteriori* error estimation, non-oscillatory k-exact reconstruction, point inversion.

1 Introduction

Aerodynamic shape optimal design [17,30,43] plays an increasingly important role in the design of vehicle and aircraft. The main components of the aerodynamic shape optimal

*Corresponding author. *Email addresses:* `xcmeng@bnu.edu.cn` (X. Meng), `garyhu@um.edu.mo` (G. Hu)

design include a flow solver, shape parameterization, optimization algorithms, and mesh deformation methods, we refer to [12, 43, 46] and the references therein for the details. To efficiently implement the aerodynamic shape optimization, one needs to develop an efficient flow solver and an appropriate shape parameterization technique.

The high order numerical methods using the goal-oriented mesh adaptation technique have been commonly used to efficiently compute the quantity of interest [6, 7, 16, 18, 21, 29, 42, 48]. On the other hand, the B-splines and Non-Uniform Rational B-splines (NURBS) [36] have been widely utilized for the shape parameterization in both the structural shape optimization problem [11, 37, 45] and the aerodynamic shape optimization problem [3, 43, 46]. Although the goal-oriented mesh adaptation technique is popular in numerical analysis, and NURBS are prevalent in Computer Aided Design (CAD), it should be pointed out that their quality combination, which should be very attractive, is not a trivial task. Hence, the primary goal of this paper is to investigate those two techniques in a synthesize way to solve the steady Euler equations by following our previous works [23, 26, 27, 35]. It is worthwhile to note that the adaptive refinement is still primarily an academic endeavor rather than an industrial technology, and the reason for this phenomenon may be that the link for the communication between the mesh refinement and the CAD system is often unavailable [28].

In [35], a NURBS-enhanced high order finite volume scheme on unstructured grids was developed to solve the two-dimensional (2D) steady Euler equations in a curved domain. Although the numerical results presented there have shown that the proposed method possesses the high order behavior, the method is unsatisfactory since uniformly refined meshes were used for the simulations. The uniform mesh refinement is not an efficient way to reduce the error and to save computational cost. For example, in the case of an inviscid subsonic flow through a Gaussian bump [47], the high order finite volume solver using uniformly refined meshes is not an efficient way to reduce the entropy error since the error around the Gaussian bump accounts for the majority of total entropy error. The benefits of high order methods are reduced when singularities are present in the solutions and the uniformly refined meshes are used [4, 38, 42, 49].

The NURBS-enhanced finite volume method [35] uses the NURBS to represent the curved boundary of physical domain, and the mesh refinement procedure does not need to communicate with the CAD system. To introduce the h-adaptive mesh refinement method in the numerical solver developed in [35], two issues need to be resolved well. The first one is how to efficiently obtain the parameter values of points lying on a NURBS curve. The second one is how to design reliable error indicators. We need the former one to flexibly insert and/or remove the grid points lying on the NURBS curve, and to efficiently obtain the quadrature information on the curved edges locating on the NURBS curve, while the latter one is not only for the efficiency of the algorithm, but also for the application of the method to shape optimization problems.

The p-adaptivity is another way to save the computational cost, and it has been studied in [41] for the Stokes flows by using the NURBS-enhanced finite element method (NEFEM) in combination with the hybridisable discontinuous Galerkin (HDG) method.