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Threshold Solutions for Nonlocal Reaction Diffusion Equations

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Abstract. We study the Cauchy problem for nonlocal reaction diffusion equations with bistable nonlinearity in 1D spatial domain and investigate the asymptotic behaviors of solutions with a one-parameter family of monotonically increasing and compactly supported initial data. We show that for small values of the parameter the corresponding solutions decay to 0, while for large values the related solutions converge to 1 uniformly on compacts. Moreover, we prove that the transition from extinction (converging to 0) to propagation (converging to 1) is sharp. Numerical results are provided to verify the theoretical results.

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1 Introduction

Reaction-diffusion equation

$$u_t = D\Delta u + ru(1-u) \tag{1.1}$$

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was first proposed by Fisher [22] and Kolmogorov *et al.* [26] in 1937 to describe the spatial spread of an advantageous allele in a given species (here D is the diffusion constant and r is the growth rate of the species).

Since then, reaction-diffusion equations have attracted considerable attention, as they have proved to give an accurate description of various natural phenomena, ranging from heat and mass transfer, chemical kinetics, population dynamics, neuroscience and biomedical processes to spatial ecology. For more details on reaction-diffusion equations and their applications, the interested reader is referred to [32] and the references therein.

In this paper, we consider the following nonlocal reaction-diffusion equation:

$$u_t = J*u - u - f(u), \quad x \in \mathbb{R}, \quad t \geq 0, \tag{1.2}$$

where

$$J*u = \int_{\mathbb{R}} J(x-y)u(t,y)\mathrm{d}y.$$

This type of equations arises in applications where there exists nonlocal interaction between the variables involved. In population dynamics models, for instance, there are many situations when there exists long-range mechanism (such as the individuals are competing for a resource which can redistribute itself, etc.) which is important in the process of evolution described (cf. [13, 23, 24]). In this context, the value of the population density u at a certain point x may depend explicitly not only on the value of the function at x but also on the values at other points: let $J(x-y)$ be the probability distribution of rates jumping from location y to location x, then $J*u$ is the rate at which individuals are arriving at position x from all other places and $-u(t,x)$ is the rate at which they are leaving location x to all other locations (cf. [19]).

A number of attempts were made to use nonlocal reaction-diffusion models similar to (1.2) in other fields such as material science [12], solid phase transitions [3, 17, 18, 28], and finite scale microstructures in nonlocal elasticity [30]. In particular, the relationship between (1.2) and lattice models

$$u_t = D\big[u(x+1,t) - 2u(x,t) + u(x-1,t)\big] - f(u) \tag{1.3}$$

has been explored in [2]. There is an extensive literature on other problems involving nonlocal terms of a different type. See, for instance, [14, 15, 28] for the continuum Ising model

$$u_t = \tanh(\beta J*u - h) - u,$$

where $\beta > 1$ and h are constants, and [4–6, 10] for nonlocal models involving the fractional Laplacian.

Throughout this paper, we make the following assumptions: