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A Lifting Method for Fokker-Planck Equations with Drift-Admitting Jumps

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Abstract. We develop in this paper a lifting method for Fokker-Planck equations with drift-admitting jumps, such that high-order finite difference schemes can be constructed directly based on grids with pure solution points. To illustrate the idea, we present as an example the construction of a fifth-order finite difference scheme. The validity of the scheme is demonstrated by conducting numerical experiments for the cases with drift admitting one jump and two jumps, respectively. Additionally, by introducing a splitting technique, we show that the lifting method can be extended to high dimensions. In particular, a two-dimensional case is studied in details to show the effectiveness of the extension.

AMS subject classifications: 65M06, 35L65, 35L81

Key words: Fokker-Planck equation, drifting-admitting jump, lifting method, finite difference method.

1 Introduction

Fokker-Planck equations with drift-admitting jumps are used as models in many science and engineering fields. For instance, they can be used to describe the dynamics of the propagators (or transition probability distributions) of some stochastic processes. To be more specific, let us consider the following one-dimensi-

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onal Langevin equation

$$\dot{v}(t)=\alpha(v)+\sqrt{2\beta}\xi(t), \tag{1.1}$$

where the dot stands for the time derivative, the drift term $\alpha(v)$ is piecewise-smooth that may contain finitely many discontinuities, and the Gaussian white noise $\xi(t)$ with strength $\beta>0$ satisfies the conditions of the zero mean $\langle\xi(t)\rangle=0$ and the correlation $\langle\xi(t)\xi(s)\rangle=\delta(t-s)$. Here δ represents the Dirac delta function. For the initial condition $v(0)=v_0$, we may denote the propagator of the process by $p(v,t|v_0,0)$, which is governed by the well-known Fokker-Planck equation

$$\partial_t p(v,t|v_0,0)=-\partial_v[\alpha(v)p]+\beta\partial_v^2 p \tag{1.2}$$

with the initial value condition

$$p(v,0|v_0,0)=\delta(v-v_0). \tag{1.3}$$

The simplest case corresponds to the so-called Brownian motion with pure dry friction [7], where the drift depends only on the sign of v. In that case, the propagator is available in closed analytic form [2, 11, 14]. For other cases with two-value drift [11, 13] or piecewise-linear drift [1, 3, 6, 14], some analytic results are also available. However, in most cases it is impractical to solve Eq. (1.2) analytically. Thus, it is necessitated to develop some effective numerical methods to investigate the underlying dynamics.

Due to the discontinuous drift $\alpha(v)$, the propagator p is nonsmooth at discontinuous points. Thus it is difficult to develop high-order numerical methods for Eq. (1.2), especially for finite difference methods. To construct an effective numerical scheme, the continuity conditions of p and the flux

$$f(v,t|v_0,0)=-\alpha(v)p+\beta\partial_v p \tag{1.4}$$

should be imposed appropriately at each discontinuous point of the drift. To the best of our knowledge, there are only a few numerical results considering the case with discontinuous drift. For the pure dry friction case studied in [12], the Fokker-Planck equation was transformed to a Schrödinger equation with a delta potential, and then studied numerically by solving a corresponding Brinkman hierarchy. Schemes with second-order convergence rate were designed for Eq. (1.2) in [15] by using an immersed interface method or in [16] by a finite volume method. By using grids staggered by flux points and solution points, a second-order finite difference scheme was developed for Eq. (1.2) in [4]. Later on, the scheme was improved to be of fifth-order accuracy in [5]. It is noted that an interpolation step from solution points to flux points is needed for the finite difference schemes developed in [4, 5].