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Orthogonal Exponentials of Planar Self-Affine Measures with Four-Element Digit Set

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Abstract. Let $\mu_{M,\mathcal{D}}$ be a self-affine measure generated by an expanding real matrix $M=\begin{pmatrix} a & e \\ f & b \end{pmatrix}$ and the digit set $\mathcal{D}=\{(0,0)^t,(1,0)^t,(0,1)^t,(1,1)^t\}$. In this paper, we consider that when does $L^2(\mu_{M,\mathcal{D}})$ admit an infinite orthogonal set of exponential functions? Moreover, we obtain that if $e=f=0$ and $a,b\in\{\frac{p}{q},p,q\in 2\mathbb{Z}+1\}$, then there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,\mathcal{D}})$, and the number 4 is the best possible.

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Key words: Infinite orthogonal set, self-affine measure, orthogonal exponentials.

1 Introduction

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$. μ is called a spectral measure if there exists a discrect set $\Lambda\subseteq\mathbb{R}^n$ such that the collection of exponential fuctions $E(\Lambda):=\{e^{2\pi i\langle\lambda,x\rangle}:\lambda\in\Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. The set Λ is then called a spectrum for μ.

The existence of a spectrum for μ is a basic problem in harmonic analysis, it was initiated by Fuglede in his seminal paper [10]. The first singularly continuous spectral measure was discovered by Jorgensen and Pedersen [12]. This surprising discovery has received a lot of attention, and the research on the spectrality of self-affine measures has become an interesting topic. Also, new spectral measures were found in [2,8,9,13,15,16] and references cited therein.

For a more general Bernoulli convolution $\mu_{\rho,N}$, Hu and Lau [11] showed that the necessary and sufficient condition that $L^2(\mu_{\rho,2})(0<\rho<1)$ contains an infinite orthogonal

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set of exponential functions is that $(\frac{p}{q})^{\frac{1}{r}}$ for some positive integers p,q,r, with p,q being odd and even, respectively. Recently, the spectral problem of Bernoulli convolution was solved by Dai [1]. He showed that $\mu_{\rho,2}$ is a spectral measure if and only if the contraction rate ρ is the reciprocal of an even integer. These results are generalized further to the N-Bernoulli measures [3,5,22].

Unlike the one dimensional case, there are few results in higher dimension. Deng and Lau [6] proved that if $M=diag(\rho,\rho)(|\rho|>1)$, with $\mathcal{D}=\{(0,0)^t,(1,0)^t,(0,1)^t\}$, then $L^2(\mu_{\rho,\mathcal{D}})$ contains an infinite orthogonal set of exponential functions if and only if $|\rho|=(\frac{p}{3q})^{\frac{1}{r}}$ for $p,q,r\in\mathbb{N}$ with $gcd(p,3q)=1$ and $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $\rho\in 3\mathbb{Z}$. Recently, for $M=diag(a,b)(a,b>1)$ and above $\mathcal{D}$, Dai, Fu and Yan [4] obtained some more general results.

For non-spectral problem of self-affine measure, Dutkay and Jorgensen [7] proved that if $M=diag(p,p)$, with $p\in\mathbb{Z}\backslash 3\mathbb{Z}$ and $\mathcal{D}=\{(0,0)^t,(1,0)^t,(0,1)^t\}$, then there are no 4 mutually orthogonal exponential functions in $L^2(\mu_{M,\mathcal{D}})$. Following this discovery, the theory of non-spectral measures has been extensively studied, The readers may see [17–19] and references cited therein.

The question of existence of orthogonal families of complex exponentials was raised, and answered, first for $dim=1$ in the case of families of Cantor constructions [12]. It turns out that, when an affine fractal is specified as a self-similar measure μ, then the answer to the question of existence of an orthogonal Fourier basis (ONB) in $L^2(\mu)$ is sensitive to choice of scaling numbers. For $dim=2$, possible Fourier bases exists when the scaling matrix satisfies specific algebraic conditions. The present paper [4, 6, 7] addresses this question for planar constructions of such measures μ. For the planar cases, there is a host of additional issues entering into the structure of Fourier bases in $L^2(\mu)$. This paper concerned with the nature of the set of Fourier frequencies, in the case when a Fourier ONB exists. Let $\mu_{M,\mathcal{D}}$ be a class of self-affine measures which satisfies

$$\mu_{M,\mathcal{D}}(\cdot)=\frac{1}{\sharp\mathcal{D}}\sum_{d\in\mathcal{D}}\mu_{M,\mathcal{D}}(M(\cdot)-d), \tag{1.1}$$

where $M=\begin{pmatrix} a & e \\ f & b \end{pmatrix}$ with $|det(M)|>1$, is an expanding matrix, and

$$\mathcal{D}=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

Our main results are the following three theorems.

Theorem 1.1. *If $e,f\neq 0$ and $a+b=0$, then $L^2(\mu_{M,\mathcal{D}})$ admits an infinite orthogonal set of exponential functions if and only if $det(M)$ is in the set $\{\pm(\frac{p}{q})^{\frac{1}{r}}: p\in 2\mathbb{Z}^+, q\in 2\mathbb{Z}^+-1, r\in\mathbb{Z}^+\}$.*

Theorem 1.2. *If $e=f=0$, i.e.$M=diag(a,b)$, then $L^2(\mu_{M,\mathcal{D}})$ admits an infinite orthogonal set of exponential functions if and only if there exist a number of a,b that are in the set $\{\pm(\frac{p}{q})^{\frac{1}{r}}:\ p\in 2\mathbb{Z}^+,\ q\in 2\mathbb{Z}^+-1,\ r\in\mathbb{Z}^+\}$.*