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A Singular Moser-Trudinger Inequality on Metric Measure Space

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Abstract. Let (X,d,μ) be a metric space with a Borel-measure μ, suppose μ satisfies the Ahlfors-regular condition, i.e.

$$b_1 r^s \leq \mu(B_r(x)) \leq b_2 r^s, \qquad \forall B_r(x) \subset X, \ r>0,$$

where b_1, b_2 are two positive constants and s is the volume growth exponent. In this paper, we mainly study two things, one is to consider the best constant of the Moser-Trudinger inequality on such metric space under the condition that s is not less than 2. The other is to study the generalized Moser-Trudinger inequality with a singular weight.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, the classical Sobolev inequalities assert that, for $1 \leq p < n$, the Sobolev space $W_0^{1,p}(\Omega)$ can be embedded into $L^q(\Omega)$, with q satisfying $1 \leq q \leq \frac{np}{n-p}$. However, it can not be embedded into $L^\infty(\Omega)$ when $p=n$. For example, $u(x) = \log|\log|x||\chi_{B_{1/e}(0)} \in W_0^{1,2}(B_{1/e}(0))$, but not bounded when $n=2$.

This gap is filled up by the Möser-Trudinger inequality [1,2], see also [3] which asserts that

$$\sup_{u\in W_0^{1,n}(\Omega),\int_\Omega |\nabla u|^n \leq 1} \int_\Omega \exp(\beta|u(x)|^{\frac{n}{n-1}}) \mathrm{d}x < \infty, \qquad \forall \beta \leq \beta_0, \tag{1.1}$$

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where $\beta_0 = n\omega_{n-1}^{\frac{1}{n-1}}$ and ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$.

This result has been generalized to n-dimensional closed manifolds with the same sharp constant β_0 [4].

Later, Adimurthi and Sandeep [5] generalized the inequality (1.1) into the following singular version,

$$\sup_{u\in W_0^{1,n}(\Omega),\int_\Omega|\nabla u|^n\le 1}\int_\Omega \frac{\exp(\beta|u|^{\frac{n}{n-1}})}{|x|^\alpha}\mathrm{d}x<\infty,$$

if and only if $\frac{\beta}{\beta_0}+\frac{\alpha}{n}\le 1$, and $\alpha\in[0,n)$, $n\ge 2$.

These inequalities are essential to solve the critical exponent equation with a singular weight, see [5–11]. In the last decades, analysis on abstract metric measure spaces have attracted extensive attention and made a great progress in this direction. See [8, 12–15] and references therein. We emphasize that various functional inequalities have been generalized to these spaces, [16]. This is also our starting point.

In this paper, we consider a generalized Möser-Trudinger inequality in metric measure space. In order to introduce our result, we first fix some notations. Let (X,d,μ) be a separable metric space with a Borel-regular measure μ, i.e. all Borel subsets are μ-measurable, and for each subset $A\subset X$, there exists a Borel set $B\subset X$ such that $A\subset B$ and $\mu(A)=\mu(B)$. We do not assume the completeness of X. Let $p\in X$ be a fixed point and B a metric ball centered at p. Denoted by $5\sigma B$ the concentric ball with radius 5σ times that of B for some constant $\sigma\ge 1$, and u_B the average of the integral of u over B, namely

$$u_B=\frac{1}{\mu(B)}\int_B u\mathrm{d}\mu.$$

The classical Möser-Trudinger inequality has been generalized to the connected metric measure space. See Hajlasz and Koskela [17].

In the sequel, we always assume $\Omega\subset X$ is a connected domain satisfying $0<\mu(\Omega)<\infty$.

Definition 1.1. *A Borel measure μ is said to be of polynomial growth [also called Ahlfors-regular] on Ω, if there exist two positive constants $b_1,b_2>0$ such that $\forall x\in\Omega$ and $r<D$, there holds*

$$b_1\mu(\Omega)(r/D)^s\le\mu(B_r(x))\le b_2\mu(\Omega)(r/D)^s, \tag{1.2}$$

where $D=\mathrm{diam}(\Omega)$ is the diameter of the domain Ω.

A direct difficulty in metric measure space is caused by the lack of gradient vector, we introduce the generalized gradient [13].

Definition 1.2. *A positive function g_Ω is said to be a generalized gradient of f on Ω, if*

$$|f(x)-f(y)|\le d(x,y)\big(g(x)+g(y)\big)$$

for all $x,y\in\Omega\setminus E$, and $E\subset\Omega$ with $\mu(E)=0$.