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On Some Properties of the Curl Operator and Their Consequences for the Navier-Stokes System

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In honor of our friend Professor Chaojiang Xu, on the occasion of his 65th birthday.

Abstract. We investigate some geometric properties of the curl operator, based on its diagonalization and its expression as a non-local symmetry of the pseudo-derivative $(-\Delta)^{1/2}$ among divergence-free vector fields with finite energy. In this context, we introduce the notion of spin-definite fields, i.e. eigenvectors of $(-\Delta)^{-1/2}$curl. The two spin-definite components of a general 3D incompressible flow untangle the right-handed motion from the left-handed one. Having observed that the non-linearity of Navier-Stokes has the structure of a cross-product and its weak (distributional) form is a determinant that involves the vorticity, the velocity and a test function, we revisit the conservation of energy and the balance of helicity in a geometrical fashion. We show that in the case of a finite-time blow-up, both spin-definite components of the flow will explode simultaneously and with equal rates, i.e. singularities in 3D are the result of a conflict of spin, which is impossible in the poorer geometry of 2D flows. We investigate the role of the local and non-local determinants

$$\int_0^T \int_{\mathbb{R}^3} \det\big(\mathrm{curl}\, u, u, (-\Delta)^{\theta} u\big)$$

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and their spin-definite counterparts, which drive the enstrophy and, more generally, are responsible for the regularity of the flow and the emergence of singularities or quasi-singularities. As such, they are at the core of turbulence phenomena.

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1 Introduction

The initial value problem for the Navier-Stokes system for incompressible fluids is usually written as

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u = -\nabla p, \quad \operatorname{div} u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1.1}$$

Here $u = u(t,x)$ is a time-dependent vector field on $\mathbb{R}^3$, the viscosity ν is a positive parameter (expressed in Stokes, i.e. $\mathtt{L}^2\mathtt{T}^{-1}$) and u_0 is a given divergence-free vector field.

In 1934, Leray [58] proved the existence of global weak solutions in $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$. In 3D, the question of their uniqueness remains elusive and is intimately connected to deciding whether the weak solutions enjoy a higher regularity. Well-posedness in various function spaces has been studied thoroughly and culminates in Koch and Tataru's result [52] if the data u_0 is small in the largest (i.e. less constraining) function space (called BMO^{-1}) that is scale and translation invariant and on which the heat flow remains locally uniformly in $L^2_{t,x}$.

The set of singular times may or not be empty, but it is a compact subset of $\mathbb{R}_+$, whose Hausdorff measure of dimension $\frac{1}{2}$ is zero. The celebrated theorem of Caffarelli *et al.* [17] ensures that singularities form a subset of space-time whose parabolic Hausdorff measure of dimension 1 vanishes too (see also Arnold and Craig [2]).

Note that Eq. (1.1) corresponds to an Eulerian point of view, i.e. it describes the movement of the fluid in a fixed reference frame. The natural question of tracking individual fluid particles, i.e. the Lagrangian point of view, is equivalent to the existence of a flow $\xi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$

$$\frac{\partial \xi}{\partial t} = u\big(t, \xi(t,x)\big), \quad \xi(0,x) = x. \tag{1.2}$$