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First Order Hardy Inequalities Revisited

Xia Huang[1] and Dong Ye[1,2,*]

[1] *School of Mathematical Sciences, East China Normal University and Shanghai Key Laboratory of PMMP, Shanghai 200241, China.*
[2] *IECL, UMR 7502, University of Lorraine, 57050 Metz, France.*

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Abstract. In this paper, we consider the first order Hardy inequalities using simple equalities. This basic setting not only permits to derive quickly many well-known Hardy inequalities with optimal constants, but also supplies improved or new estimates in miscellaneous situations, such as multipolar potential, the exponential weight, hyperbolic space, Heisenberg group, the edge Laplacian, or the Grushin type operator.

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1 Introduction

The Hardy inequalities go back to G.H. Hardy, who showed in [23] a very famous estimate: Let $p>1$, then

$$\int_0^\infty |u'(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_0^\infty \frac{|u(x)|^p}{x^p} dx, \quad \forall u \in C^1(\mathbb{R}_+), \quad u(0)=0. \tag{1.1}$$

Since one century, the Hardy type inequalities have been enriched extensively and broadly, they play important roles in many branches of analysis and geometry. More generally, we call first order Hardy inequalities, the estimates like

*Corresponding author. *Email addresses:* `xhuang@cpde.ecnu.edu.cn` (X. Huang), `dye@math.ecnu.edu.cn` (D. Ye)

$$\int_{\mathcal{M}} V|\nabla u|^p d\mu \geq \int_{\mathcal{M}} W|u|^p d\mu$$

with positive weights V, W and u in suitable function spaces. They are also called weighted Poincaré or Hardy-Poincaré inequalities.

A huge literature exists on the Hardy inequalities, it is just impossible nor our intension to mention all the progress even for the first order case, we refer to the classical and recent books [2,22,28,31,32] for interested readers. The modest objective here is to show that many first order Hardy inequalities can be derived naturally and quickly from simple equalities, which on one hand yield classical Hardy inequalities with optimal constants, and on the other hand provide new Hardy inequalities or improve some well-known results.

The most well-known first order Hardy inequality is the following: Let $\alpha \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^\alpha} dx \geq \frac{(n-2-\alpha)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^{\alpha+2}} dx, \quad \forall u \in C_c^1\left(\mathbb{R}^n \backslash \{0\}\right). \tag{1.2}$$

Here and after, $|\cdot|$ denotes the Euclidean norm. The optimal constant could be firstly shown in [24, p. 259] with $n=1$. For simplicity, we consider only real valued functions, and without special remark, the functions u are C^1, compactly supported away from the singularities of involved weights. In general, applying density argument, many estimates hold true in larger functional spaces.

The inequality (1.2) can be seen as a direct consequence of the following equality (see for instance [16] with $\alpha=0$ and [33, Lemma 2.3(i)] with $\alpha=2$). For any $u \in C_c^1(\mathbb{R}^n \backslash \{0\})$ and $\alpha \in \mathbb{R}$, if $v = |x|^{\frac{n-2-\alpha}{2}} u$, there holds

$$\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^\alpha} dx = \frac{(n-2-\alpha)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^{\alpha+2}} dx + \int_{\mathbb{R}^n} \frac{|\nabla v|^2}{|x|^{n-2}} dx. \tag{1.3}$$

There is also a radial derivative version of the above equality: $\forall u \in C_c^1(\mathbb{R}^n \backslash \{0\})$,

$$\int_{\mathbb{R}^n} \frac{|\partial_r u|^2}{|x|^\alpha} dx = \frac{(n-2-\alpha)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^{\alpha+2}} dx + \int_{\mathbb{R}^n} \frac{|T_\alpha(u)|^2}{|x|^\alpha} dx, \tag{1.4}$$

where ∂_r is the radial derivative $\partial_r = \frac{x \cdot \nabla}{r}$, and $T_\alpha = \partial_r + \frac{n-2-\alpha}{2r}$. Indeed, the equalities (1.3) and (1.4) are equivalent, since

$$|\nabla u|^2 = |\partial_r u|^2 + \sum_{j=1}^n |L_j u|^2,$$

where

$$L_j u = \partial_j u - \frac{x_j}{r} \partial_r u, \quad \forall 1 \leq j \leq n.$$