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Global-in-Time L^p-L^q Estimates for Solutions of the Kramers-Fokker-Planck Equation

Xue Ping Wang[1,*] and Lu Zhu[2]

[1] *Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629, Université de Nantes, 44322 Nantes Cedex 3, France.*
[2] *College of Science, Hohai University, 210024 Nanjing, China.*

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Dedicated to the 65-th birthday of Professor Chaojiang Xu

Abstract. In this work, we prove an optimal global-in-time L^p-L^q estimate for solutions to the Kramers-Fokker-Planck equation with short range potential in dimension three. Our result shows that the decay rate as $t\to+\infty$ is the same as the heat equation in x-variables and the divergence rate as $t\to 0_+$ is related to the sub-ellipticity with loss of one third derivatives of the Kramers-Fokker-Planck operator.

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1 Introduction

The Kramers-Fokker-Planck equation is the evolution equation for the distribution functions describing the Brownian motion of particles in an external field

$$\frac{\partial W}{\partial t}=\left(-v\cdot\nabla_x+\nabla_v\cdot\left(\gamma v-\frac{F(x)}{m}\right)+\frac{\gamma kT}{m}\Delta_v\right)W, \tag{1.1}$$

*Corresponding author. *Email addresses:* `xue-ping.wang@univ-nantes.fr` (X.P. Wang), `zhulu@hhu.edu.cn` (L. Zhu)

where $F(x)=-m\nabla V(x)$ is the external force and $W=W(t;x,v)$ is the distribution function of particles for $x,v\in\mathbb{R}^n$ and $t>0$. In this equation, x and v represent the position and velocity variables of particles, m the mass, k the Boltzmann constant, γ the friction coefficient and T the temperature of the media. This equation, called the Kramers equation in the book of Risken [14], was initially derived and used by Kramers [8] to describe kinetics of chemical reaction. Later on it turned out that it had more general applicability to different fields such as supersonic conductors, Josephson tunneling junction and relaxation of dipoles. Eq. (1.1), also often called the Fokker-Planck equation, is in fact a special case of the more general Fokker-Planck equation [14] or the Kolmogorov forward equation for continuous-time diffusion processes [7].

After appropriate normalization of physical constants and change of the unknown, the KFP equation can be written into the form

$$\partial_t u(t;x,v)+Pu(t;x,v)=0, \quad (x,v)\in\mathbb{R}^n\times\mathbb{R}^n, \quad t>0 \tag{1.2}$$

with initial data

$$u(0;x,v)=u_0(x,v), \tag{1.3}$$

where P is the KFP operator defined by

$$P=-\Delta_v+\frac{1}{4}|v|^2-\frac{n}{2}+v\cdot\nabla_x-\nabla V(x)\cdot\nabla_v. \tag{1.4}$$

In this work, $V(x)$ is supposed to be a real-valued C^1 function verifying

$$|V(x)|+\langle x\rangle|\nabla V(x)|\le C\langle x\rangle^{-\rho}, \quad x\in\mathbb{R}^n \tag{1.5}$$

for some $\rho>-1$. Here $\langle x\rangle=(1+|x|^2)^{\frac{1}{2}}$. Remark that $V(x)$ is determined up to an additive constant. Eq. (1.5) implies that when with $\rho>0$, this constant is chosen such that

$$\lim_{|x|\to\infty} V(x)=0,$$

which can be interpreted as a normalization condition for $V(x)$. Let $\mathfrak{m}$ be the function defined by

$$\mathfrak{m}(x,v)=\frac{1}{(2\pi)^{\frac{n}{4}}}e^{-\frac{1}{2}(\frac{v^2}{2}+V(x))}. \tag{1.6}$$

Then $\mathfrak{M}=\mathfrak{m}^2$ is the Maxwellian [14] and $\mathfrak{m}$ verifies the stationary KFP equation

$$P\mathfrak{m}=0 \quad \text{in } \mathbb{R}^{2n}_{x,v}. \tag{1.7}$$