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Traveling Waves in Degenerate Diffusion Equations

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Abstract. In this survey, we present a literature review on the study of traveling waves in degenerate diffusion equations by illustrating the interesting and singular wave behavior caused by degeneracy. The main results on wave existence and stability are presented for the typical degenerate equations, including porous medium equations, flux limited diffusion equations, delayed degenerate diffusion equations, and other strong degenerate diffusion equations.

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1 Introduction

Degenerate diffusion models appear in the theory of many biological and physical applications, such as population spreading, tumor invasions, fluid dynamics, and many practical problems [39, 41, 50, 51, 54]. To be precise, flow of an ideal gas through a porous medium can be described by the porous medium equation [54]; flux-saturated equations were introduced in the inertial confinement fusion [20]. In this review, we will present some of the recent developments of traveling waves related to nonlinear diffusion equations.

The mathematical interest in degenerate diffusion equations is now steadily growing. Fundamental features of degenerate diffusion equations account for the

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propagation properties of the solutions such as the finite propagation property and the waiting time phenomenon. Finite propagation means the persistence of compact supports of solutions, while waiting time phenomenon represents that the support of the solution remains motionless for some time before starting to move. Another fundamental property of this type of equations is the existence of discontinuous traveling waves in strong degenerate diffusion equations.

Traveling waves, i.e., solutions of the type $u(t,x)=u(x-ct)$, starts from nonlinear reaction-diffusion equation (with linear diffusion) of the type

$$\frac{\partial u}{\partial t}=D\Delta u+f(u),$$

where D is the diffusion coefficient and $f(u)$ is the nonlinear reaction function. Since the pioneering works of Fisher [24] and Kolmogorov *et al.* [32] in the 30s, traveling waves describe various dynamical behaviors in propagation processes. Existence and stability of traveling waves for nonlinear reaction-diffusion equations (with linear diffusion) have been widely studied in the literature, see for example, [16,22,23,29,36,42,43] and the references therein. In particular, the degenerate or singular diffusion equations have also been investigated in a series of works [4,5,11,12].

In the following sections, we will first discuss the traveling waves for the several typical degenerate diffusion equations, including the classical porous medium equations, flux limited diffusion equations, delayed degenerate diffusion equations. Finally, we will also present traveling waves for some other strong degenerate diffusion equations, especially the corresponding sharp discontinuous traveling waves.

2 Traveling waves in porous medium equations

Porous medium equations arising in many biological and physical systems are one of the typical degenerate diffusion equations. In population dynamics model, positive density-dependent dispersal implies that the population migrates to regions of lower density more rapidly as the population gets more crowded. The population dispersal can be influenced by many biological, chemical and physical factors, such as overcrowding, social behavior, mating, lights, food, chemical substances, etc [53]. The derivation of porous medium type density-dependent diffusion implies that the population jump probability is a density-dependent function due to the conspecific repelling or self-organization [49,55].

For porous media equation, there is a well-established theory on the traveling waves. To appreciate this analysis, consider the following reaction diffusion