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Fujita-Kato Theorem for the Inhomogeneous Incompressible Navier-Stokes Equations with Nonnegative Density

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Abstract. In this paper, we prove the global existence and uniqueness of solutions for the inhomogeneous Navier-Stokes equations with the initial data $(\rho_0,u_0)\in L^\infty\times H_0^s$, $s>\frac{1}{2}$ and $\|u_0\|_{H_0^s}\le\varepsilon_0$ in bounded domain $\Omega\subset\mathbb{R}^3$, in which the density is assumed to be nonnegative. The regularity of initial data is weaker than the previous $(\rho_0,u_0)\in(W^{1,\gamma}\cap L^\infty)\times H_0^1$ in [13] and $(\rho_0,u_0)\in L^\infty\times H_0^1$ in [7], which constitutes a positive answer to the question raised by Danchin and Mucha in [7]. The methods used in this paper are mainly the classical time weighted energy estimate and Lagrangian approach, and the continuity argument and shift of integrability method are applied to complete our proof.

AMS subject classifications: 35Q30, 76D03

Key words: Inhomogeneous Navier-Stokes equations, nonnegative density, global existence and uniqueness.

1 Introduction

In this work, we will investigate the global existence and uniqueness of solutions to the following inhomogeneous incompressible Navier-Stokes equations:

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$$\begin{cases} \rho_t+u\cdot\nabla\rho=0 & \text{in } \mathbb{R}^+\times\Omega, & (1.1\text{a})\\ \rho u_t+\rho u\cdot\nabla u-\Delta u+\nabla P=0 & \text{in } \mathbb{R}^+\times\Omega, & (1.1\text{b})\\ \operatorname{div} u=0 & \text{in } \mathbb{R}^+\times\Omega & (1.1\text{c}) \end{cases}$$

with the initial and the boundary conditions

$$\begin{aligned} &(\rho,\rho u)|_{t=0}=(\rho_0,\rho_0 u_0),\\ &u(t,x)=0, \quad x\in\partial\Omega, \end{aligned}$$

and ρ,u,P standing for the density, velocity and pressure respectively. This system originated from the theory of geophysical flows, which describes incompressible fluids with different densities, and is also widely used in the research of two miscible fluids. One may check [14] for the detailed derivation of this system.

The weak solution theory of this system (see [14]) is well known. If

$$0\le\rho_0\le\rho^* \text{ for some } \rho^*>0 \text{ and } \sqrt{\rho_0}u_0\in L^2,$$

there exists global weak solutions (ρ,u) to system (1.1) such that for all $t\ge0$

$$\begin{aligned} &\left\|\sqrt{\rho(t)}u(t)\right\|_{L^2}^2+2\int_0^t\|\nabla u\|_{L^2}^2 d\tau\le\|\sqrt{\rho_0}u_0\|_{L^2}^2,\\ &0\le\rho(t)\le\rho^* \text{ and } \int_\Omega\rho dx=\int_\Omega\rho_0 dx. \end{aligned}$$

However, the uniqueness of such weak solutions is still an open problem. As far as we know, Ladyzhenskaya and Solonnikov [12] firstly proved the global well-posedness of system (1.1) with the small data and the density bounded away from zero. After that, many classical results appeared.

When the density is a constant, the system (1.1) is reduced to the incompressible Navier-Stokes equations

$$\begin{cases} u_t+u\cdot\nabla u-\Delta u+\nabla P=0,\\ \operatorname{div} u=0. \end{cases} \tag{1.2}$$

In the seminal paper [9], Fujita and Kato proved the global well-posedness for the initial data $u_0\in\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ satisfying $\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}\le\varepsilon_0$ with small $\varepsilon_0>0$. It is well known that the space $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ is critical for its important property that the norm is invariant under the scaling of the equation

$$u(t,x)\to\lambda u(\lambda^2 t,\lambda x),$$

which means that if $u(t,x)$ is a solution of (1.2), then so does $u_\lambda\triangleq\lambda u(\lambda^2 t,\lambda x)$.