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Preface

Special Issues on Nonlocal Modeling, Analysis, and Computation

Nonlocal models have been established as extremely powerful modeling tools. The nonlocal operators involved in those models have shown excellent capability to describe the effects of long-range interactions in space and historical memories in time, see e.g., [3, 14, 18, 43]. The spectrum of their practical applications is very broad and spans many diverse disciplines, e.g., fracture mechanics, subsurface flow, image processing, data analysis, thermal diffusion in fractal domains, and dynamics of protein molecules. Compared with classical local partial differential equations (PDE), nonlocal models pose substantial challenges to both their theory and numerical analysis. In the last two decades, intensive research has been carried out in this area. See [1, 4, 9, 14, 17–19, 29, 31, 41, 53] for a rather incomplete list of monographs and review papers.

This special issue consists of thirteen invited contributions on various aspects of nonlocal models. These papers cover a broad range of topics including theoretical analysis of novel nonlocal models, construction of computational schemes, stability and error analysis, and numerical simulations and validations. Below we briefly describe the content of the special issue.

The papers [15, 23, 27] present mathematical analysis of some novel nonlocal models. In [15], the authors study an anisotropic nonlocal diffusion model, where the nonlocal diffusion operator involves an anisotropic tensor. The wellposedness of the model is proved by establishing an equivalence between weighted and unweighted anisotropic nonlocal diffusion operators. This paper extends the nonlocal calculus developed in early works [16, 19, 20]. Moreover, the authors also extend the argument to the anisotropic diffusion-advection equation describing anomalous transport of solutes.

Nonlocal Stokes equations with periodic boundary conditions were developed and analyzed in pioneer works [24, 33]. The paper [23] introduces a novel nonlocal Stokes system with a physical no-slip boundary condition. Such a kind of Dirichlet type boundary condition is enforced by using the idea of volume constraint. With an additional nonlocal relaxation term, which disappears as the nonlocality vanishes, the stability and well-posedness are proved where smooth kernel functions are used to define the nonlocal operators. Moreover, the authors show that the vanishing nonlocality limit of the solution to the nonlocal system is that to the classical Stokes system.

The Gierer-Meinhardt model [27] is a prototypical activator-inhibitor reaction-diffusion system [54]. While the original model has assumed a normal diffusion process, a growing body of literature has considered the alternative of anomalous diffusion which may be better suited for biological processes in complex environments. The paper [28] deals with a one-dimensional Gierer–Meinhardt model involving a fractional Laplacian with periodic boundary conditions. The existence and stability of multi-spike solutions are rigorously examined by using a Lyapunov-Schmidt type reduction and the method of matched asymptotic expansions.

The papers [2, 5, 35, 57] deal with the numerical treatment for nonlocal-in-space models. In particular, a dual-horizon nonlocal diffusion model is considered in [2]. In such a model, the influence area at each point consists of a standard sphere horizon and an irregular dual horizon whose geometry is determined by the distribution of the varying horizon parameter. The well-posedness as well as some useful properties, such as mass conservation and maximum principle, are theoretically examined. The authors propose a Galerkin finite element discretization, which is known to be asymptotically compatible [51, 52], and introduce its implementation in details. Various numerical experiments in high dimensions are presented to illustrate the usage of the variable horizon and demonstrate the convergence of the numerical scheme.

On bounded domains, the spectral fractional Laplace equation can be solved efficiently via Balakrishnan integral together with suitable quadrature rules [6–8]. The paper [5] presents an numerical scheme for solving spectral fractional Laplace-Beltrami problems on a closed surface. The proposed algorithm relies on the aforementioned integral representation and applies a SINC quadrature coupled with standard Galerkin finite element methods for parametric surfaces. Optimal convergence rates (up to a log factor) are derived analytically and observed numerically in L^2 and H^1.

The papers [35, 57] study spectral methods for nonlocal models. Since the solutions to nonlocal problems usually contain weak singularities even for smooth problem data, the classical spectral methods using polynomials generally do not work well. One promising idea is to use nonpolynomial basis functions [11,56]. The paper [57] developed and analyzed a hybrid spectral method for nonlinear Volterra integral equations (with a weakly singular kernel) of the second kind. The key idea is to use the shifted generalized Log orthogonal functions [10, 12] as the basis in the boundary intervals.

Nonlocal diffusion equations in unbounded domains attract a lot of attention in recent years [42,45,48,50]. The paper [35] develops and analyzes an efficient Hermite spectral-Galerkin method for solving the nonlocal diffusion equations in $\mathbb{R}^d$ with $d = 1, 2$. The use of Hermite functions can analytically de-convolute the inner integral, and the singularity of the kernel functions can be absorbed in the computation. Detailed implementation, rigorous error analysis and numerical examples are provided.

The list of applications of nonlocal operators is ever increasing. The paper [44] develops a novel edge detection method of the grayscale image, by using a nonlocal Laplacian operator. This novel strategy is applied to setup the initial value of an Allen-Cahn equation. Then the two-phase segmentation of the grayscale image could be obtained by solving the Allen-Cahn equation with exponential time differencing schemes [21, 22]. The authors demonstrate the effectiveness and efficiency of the proposed method by plenty of numerical experiments.

The papers [13,34,36,39,46] deal with the nonlocal models involving a fractionalorder derivative in time. In particular, the papers [34, 39, 46] discuss the piecewise linear polynomial interpolation method, so-called L1 scheme, for the time discretization. The L1 scheme is one of the most popular schemes for time-fractional models, and there have been several numerical analyses under different conditions about the solution or problem data [30,32,37,40,47,55]. The paper [39] analyzes the L1 scheme with variable time step sizes for time-fractional Allen-Cahn equations [25,49] by using the fractional Gronwall's inequality [38]. An adaptive time-stepping strategy according to the dynamical feature of the system is presented to capture the multi-scale behaviors and to improve the computational performance. The paper [34] considers the time-fractional Stokes equation and its numerical treatment. The L1 scheme is used to discretize the time fractional derivative and the local discontinuous Galerkin method is used to discretize in space. A suboptimal error estimate is established under some conditions on the solution regularity. The paper [46] provides a comprehensive survey on the recent progress of the error analysis for the L1 scheme and its variants. Various aspects of these numerical analyses are outlined in [46], such as global and local convergence estimate, fast algorithm, semilinear problems, multi-term time derivatives, α-robustness and a posteriori error analysis.

Random effects arise naturally in physical systems. The paper [36] studies a semilinear stochastic space-time fractional wave equations driven by infinite dimensional multiplicative Gaussian noise and additive fractional Gaussian noise. The discretization of random noise results in a regularized stochastic fractional wave equation while introducing a modeling error in the mean-square sense. The solution theories of the regularized equation is established and Galerkin finite element approximation is developed with a rigorous error analysis.

Spectral deferred correction (SDC) method was first introduced in [26] to construct high-order stable methods for solving ordinary differential equations. The paper [13] discusses the SDC method for solving fractional ordinary differential equations. In particular, the SDC method is used to discretize the fractional derivative, and convergence of the SDC iteration is accelerated by applying the generalized minimal residual (GMRES) algorithm with restart. The authors demonstrate the effectiveness of the proposed method by presenting plenty of numerical examples for both stiff and non-stiff fractional models.

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Jiang Yang
Department of Mathematics, Southern University of Science and Technology,
Shenzhen, China

Zhi Zhou
Department of Applied Mathematics, The Hong Kong Polytechnic University,
Hong Kong, SAR, China

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