Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2022-0001s

Vol. **15**, No. 4, pp. 851-875
November 2022

Analysis of Anisotropic Nonlocal Diffusion Models: Well-Posedness of Fractional Problems for Anomalous Transport

Marta D'Elia* and Mamikon Gulian

Data Science and Computing, Sandia National Laboratories, CA, USA

Received 26 December 2020; Accepted (in revised version) 25 November 2021

Abstract. We analyze the well-posedness of an anisotropic, nonlocal diffusion equation. Establishing an equivalence between weighted and unweighted anisotropic nonlocal diffusion operators in the vein of unified nonlocal vector calculus, we apply our analysis to a class of fractional-order operators and present rigorous estimates for the solution of the corresponding anisotropic anomalous diffusion equation. Furthermore, we extend our analysis to the anisotropic diffusion-advection equation and prove well-posedness for fractional orders $s \in [0.5, 1)$. We also present an application of the advection-diffusion equation to anomalous transport of solutes.

AMS subject classifications: 4B10, 35R11, 35B40, 26B12

Key words: Nonlocal models, fractional models, anomalous diffusion, anisotropic diffusion, solute transport.

---

1. Introduction

Nonlocal models have become a preferred modeling choice for scientific and engineering applications featuring global behavior that is affected by small scales. In particular, nonlocal models can capture effects that classical partial differential equations (PDEs) fail to describe; these effects include multiscale behavior and anomalous transport such as superdiffusion and subdiffusion. A nonlocal equation is characterized by integral operators acting on a lengthscale or "horizon", describing long-range forces and reducing the regularity requirements on the solutions. Engineering applications include surface or subsurface transport [4, 5, 22, 32, 33, 52, 53], fracture mechanics [31,36,57], turbulence [34,46], image processing [1,12,29,37] and stochastic processes [7, 14, 42, 44, 45].

*Corresponding author. *Email addresses:* `mdelia@sandia.gov` (M. D'Elia), `mgulian@sandia.gov` (M. Gulian)

In this work we focus mainly on nonlocal operators of fractional type, of the form

$$\mathcal{L}_{\omega;A} = \mathcal{D}_\omega\big(A(\boldsymbol{x})\mathcal{G}_\omega\big) \quad \text{with} \quad \omega \propto |\boldsymbol{x}-\boldsymbol{y}|^{-(n+s)} \tag{1.1}$$

and their use in models for solute transport. As reviewed carefully in Section 2, $A(\boldsymbol{x})$ denotes a diffusion tensor and $\mathcal{D}_\omega$ and $\mathcal{G}_\omega$ denote weighted nonlocal divergence and gradient operators, respectively, with weight function ω. Modeling and simulation of surface and subsurface transport is challenging due to the inevitable heterogeneities of the media which generate, at the continuum scale, diffusion processes that exhibit transport rates which may be faster or slower than those described by the classical integer-order diffusion equation. While at the smaller scales a local PDE model may be able to accurately describe diffusion processes by explicitly embedding the heterogeneities in the model parameters, at the continuum scale, such models may fail to do so. In contrast, a fractional-order model using an diffusion operator of the form (1.1) may act as a homogenized model that encodes the heterogeneities of the medium in the integral operator itself. Several fractional models have been proposed in the literature for such applications; we refer to [59] for an extensive review. The problems we study in the present work serve as a novel models for anisotropic, anomalous transport.

Though significant progress has been made in the analysis and simulation of nonlocal equations, many modeling and computational challenges still remain. Among those, we mention expensive computational cost [2,10,13,20,49,58,61], the identification of optimal model parameters or kernel functions [8,18,19,30,47,48,62–64], and gaps and open questions within the nonlocal and fractional calculus theories [15, 54–56]. One focus of this work is on such theoretical gaps in the analysis of the so-called weighted nonlocal operators, their connection to fractional operators and on the well-posedness of the corresponding diffusion problem. The connection between nonlocal and fractional operators, investigated for the first time in [11, 17], was studied extensively in the recent work [16]. There, the authors introduce the notion of a unified nonlocal vector calculus for scalar functions and introduce a universal nonlocal Laplace operator that includes, as a special cases, the well-known fractional Laplace operators and other variants of the latter. Variational results of the unified calculus allow the extension of the well-established theory of unweighted operators [24] to weighted nonlocal operators and, more specifically, operators of fractional-order vector calculus.

We continue this effort by providing conditions for well-posedness for problems involving anisotropic nonlocal weighted operators, with a special focus on fractional operators of the form (1.1) for which we establish well-posedness.

Our major contributions are:

1. The extension of results presented in [16] to the parabolic case.

2. The well-posedness analysis of the elliptic and parabolic anisotropic nonlocal diffusion problem and, for the fractional case, the well-posedness of such problems for fractional orders $s \in (0, 1)$.