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A Nonlocal Stokes System with Volume Constraints

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Abstract. In this paper, we introduce a nonlocal model for linear steady Stokes system with physical no-slip boundary condition. We use the idea of volume constraint to enforce the no-slip boundary condition and prove that the nonlocal model is well-posed. We also show that and the solution of the nonlocal system converges to the solution of the original Stokes system as the nonlocality vanishes.

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1. Introduction

Recently, nonlocal models and corresponding numerical methods have attracted much attention due to many successful applications. For example, in solid mechanics, the theory of peridynamics [38] has been used as a possible alternative to conventional models of elasticity and fracture mechanics. Many numerical methods have also been developed to simulate nonlocal models like peridynamics based on rigorous mathematical analysis [10–12, 30, 31, 39, 43]. Nonlocal methods are also successfully applied in image processing and data analysis [2, 4, 6, 19, 20, 22, 23, 29, 33–35, 41]. The idea of integral approximation is also applied to derive numerical scheme for solving PDEs on point cloud [25, 26].

In this paper, we study the nonlocal analog of the Stokes system in fluid mechanics. Previously, nonlocal Stokes models have been proposed in [13,24] and analyzed subject to periodic boundary condition. In this paper, we consider the case of a nonlocal no-slip

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boundary condition. More precisely, for the conventional, local linear Stokes system on a domain $\Omega \subset \mathbb{R}^n$,

$$\begin{cases} \Delta \boldsymbol{u}(\boldsymbol{x}) - \nabla p(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega \end{cases} \tag{1.1}$$

the no-slip boundary condition on the boundary $\partial\Omega$ is

$$\boldsymbol{u} = 0 \quad \text{at } \partial\Omega. \tag{1.2}$$

For the pressure, we impose average zero condition

$$\int_\Omega p(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0. \tag{1.3}$$

The no-slip boundary condition is a Dirichlet type boundary condition and it is often used in many real world applications. However, the theoretical study with no-slip boundary condition is also much more difficult. The first question is how to enforce no-slip boundary condition in the nonlocal approach. Recently, Du *et al.* [10] proposed volume constraint to deal with the boundary condition in the nonlocal diffusion problem by enforcing the condition over a nonlocal region adjacent to the boundary. Adopting this idea, in the nonlocal Stokes system, we extend the no-slip condition to a small layer as shown in Fig. 1.

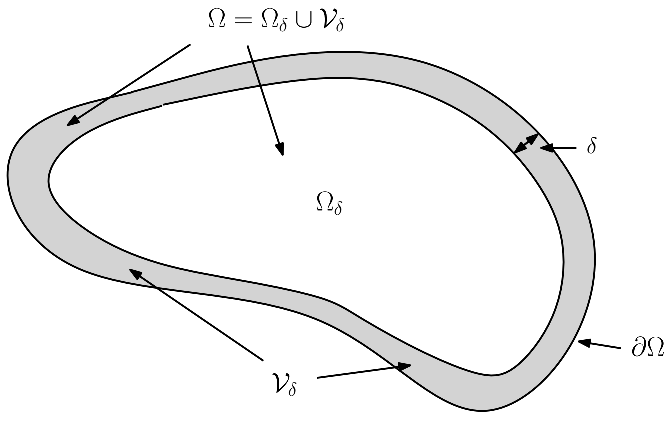

Figure 1: Computational domain in non-local Stokes model.

For a nonlocal problem involving nonlocal interactions on the range of $\delta > 0$, the whole computational domain Ω is decomposed to two parts. $\Omega = \mathcal{V}_\delta \bigcup \Omega_\delta$ as shown in Fig. 1 and $\boldsymbol{u}$ is enforced to be zero in $\mathcal{V}_\delta$, i.e.

$$\boldsymbol{u}_\delta(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathcal{V}_\delta. \tag{1.4}$$

Definition of Ω_δ and $\mathcal{V}_\delta$ will be given in (2.1). The parameter δ is often called the nonlocal horizon parameter [9, 38]. In Ω_δ, the Stokes equation is approximated is