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Stability of Spike Solutions to the Fractional Gierer-Meinhardt System in a One-Dimensional Domain

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Abstract. In this paper we consider the existence and stability of multi-spike solutions to the fractional Gierer-Meinhardt model with periodic boundary conditions. In particular we rigorously prove the existence of symmetric and asymmetric two-spike solutions using a Lyapunov-Schmidt reduction. The linear stability of these two-spike solutions is then rigorously analyzed and found to be determined by the eigenvalues of a certain 2×2 matrix. Our rigorous results are complemented by formal calculations of N-spike solutions using the method of matched asymptotic expansions. In addition, we explicitly consider examples of one- and two-spike solutions for which we numerically calculate their relevant existence and stability thresholds. By considering a one-spike solution we determine that the introduction of fractional diffusion for the activator or inhibitor will respectively destabilize or stabilize a single spike solution with respect to oscillatory instabilities. Furthermore, when considering two-spike solutions we find that the range of parameter values for which asymmetric two-spike solutions exist and for which symmetric two-spike solutions are stable with respect to competition instabilities is expanded with the introduction of fractional inhibitor diffusivity. However our calculations indicate that asymmetric two-spike solutions are always linearly unstable.

AMS subject classifications: 35R11, 35B32, 60K50, 35B25

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1. Introduction

The Gierer-Meinhardt (GM) model is a prototypical activator-inhibitor reaction-

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diffusion system that has, since its introduction by Gierer and Meinhardt in 1972 [5], been the focus of numerous mathematical studies. In the singularly perturbed limit for which the activator has an asymptotically small diffusivity the GM model is known to exhibit localized solutions in which the activator concentrates at a discrete collection of points and is otherwise exponentially small. The analysis, both rigorous and formal, of the existence, structure, and linear stability of such localized solutions has been the focus of numerous studies over the last two decades (see the book [32]). The GM model in a one-dimensional domain has been particularly well studied using both rigorous PDE methods [28, 31] as well as formal asymptotic methods [13, 26]. More recent extensions to the classical one-dimensional GM model have considered the effects of precursors [15,34], bulk-membrane-coupling [7], and anomalous diffusion [20,21,33]. It is the latter of these extensions which motivates the following paper which focuses on extending the results obtained in [20, 33] for the fractional one-dimensional GM model.

The analysis of localized solutions to the GM model fits more broadly into the study of pattern formation in reaction-diffusion systems. Such reaction-diffusion systems have widespread applicability in the modelling of biological phenomena for which distinct agents diffuse while simultaneously undergoing prescribed reaction kinetics (see the classic textbook by Murray [19]). While these models have typically assumed a normal (or Brownian) diffusion process for which the mean-squared-displacement (MSD) is proportional to the elapsed time, a growing body of literature has considered the alternative of anomalous diffusion which may be better suited for biological processes in complex environments [18, 23, 24] (see also [1, Section 7.1]). In contrast to normal diffusion, for anomalous diffusion the MSD and time are related by the power law MSD $\propto$ $(\text{time})^{\alpha}$ where an exponent satisfying $\alpha > 1$ or $\alpha < 1$ corresponds to superdiffusion or subdiffusion respectively. Studies of reaction-diffusion systems with subdiffusion and superdiffusion suggest that anomalous diffusion can have a pronounced impact on pattern formation (see [14] as well as [6] and the references therein). In particular studies have shown that both superdiffusion and subdiffusion can reduce the threshold for Turing instabilities when compared to the same systems with normal diffusion [6, 11]. Likewise it has been shown that the Hopf bifurcation threshold for spike solutions to the GM model with normal diffusion for the inhibitor and superdiffusion, mainly with Lévy flights, for the activator is decreased [20] whereas it is increased in the case of subdiffusion for the inhibitor and normal diffusion for the activator [21].

In this paper we consider the existence and stability of localized multi-spike solutions to the periodic one-dimensional GM model with Lévy flights for both the activator and the inhibitor. In particular we consider the fractional Gierer-Meinhardt system with periodic boundary conditions

$$\begin{cases} u_t + \varepsilon^{2s_1}(-\Delta)^{s_1} u + u - \dfrac{u^2}{v} = 0 & \text{for} \quad x \in (-1,1), \\ \tau v_t + D(-\Delta)^{s_2} v + v - u^2 = 0 & \text{for} \quad x \in (-1,1), \\ u(x) = u(x+2), \ \ v(x) = v(x+2) & \text{for} \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$