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L1/Local Discontinuous Galerkin Method for the Time-Fractional Stokes Equation

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Abstract. In this paper, L1/local discontinuous Galerkin method seeking the numerical solution to the time-fractional Stokes equation is displayed, where the time-fractional derivative is in the sense of Caputo with derivative order $\alpha \in (0,1)$. Although the time-fractional derivative is used, its solution may be smooth since such examples can be easily constructed. In this case, we use the uniform L1 scheme to approach the temporal derivative and use the local discontinuous Galerkin (LDG) method to approximate the spatial derivative. If the solution has a certain weak regularity at the initial time, we use the non-uniform L1 scheme to discretize the time derivative and still apply LDG method to discretizing the spatial derivative. The numerical stability and error analysis for both situations are studied. Numerical experiments are also presented which support the theoretical analysis.

AMS subject classifications: 35R11, 65M60

Key words: L1 scheme, local discontinuous Galerkin method (LDG), time-fractional Stokes equation, Caputo derivative.

1. Introduction

The time-fractional Navier-Stokes equation was considered in [6]. Shortly after, Momani and Odibat [21] used the Adomian decomposition method for solving time-fractional Navier-Stokes equation in a tube. In 2015, the well-posedness of its mild solution was investigated [1]. Very recently, the mixed finite element method and the spectral method for this time-fractional equation were explored [17,30,31]. In this paper, we numerically study a special case of the time-fractional Navier-Stokes equation, i.e., the non-stationary 2D time-fractional Stokes equations. This model characterizes the long memory processes, thus it can be used to model anomalous diffusion in fractal media [32]. Recently, Xu *et al.* [19] proved the existence and uniqueness of the

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weak solution with the help of classical saddle-point theory. Then they constructed an efficient spectral method for numerical approximations of the weak solution. Li *et al.* [16] proposed two marker and cell schemes for the time fractional Stokes equation on non-uniform grids. Based on the fixed-point theorem, Xu *et al.* [28] studied the well-posedness of a time fractional stochastic Stokes model with finite and infinite delay and multiplicative Brownian motion. Motivated by above considerations, the objective in this paper is to present an L1/LDG method for the two-dimensional time-fractional Stokes equations. LDG method is a special class of discontinuous Galerkin (DG) methods, introduced firstly by Cockburn and Shu [4]. The main technique of LDG method is to rewrite higher-order derivative equation into an equivalent system containing only the first derivative, and then discretize it by the standard DG method. This method has received increasing interest due to its features: 1) easily handling meshes with hanging nodes, elements of general shapes, and local spaces of different types [29]; 2) freely designing numerical fluxes in herited conservation laws [3]. Till now, LDG methods have been successfully applied to fractional differential equations [5, 10, 11, 14, 27] for one space dimension, [12] for both one and two space dimensions, [13] for two space dimensions. Inspired by the analysis techniques in [13, 24, 26], we study the time-fractional Stokes equation in two spatial dimensions using the L1/LDG method. The research contents include two parts:

1. If the solution is smooth enough with respect to time, for example, belonging to $C^2[0,T]$ for a given time T. This case can exist since one can construct such a solution. In this case, we use the uniform L1 scheme to compute the time-fractional derivative and apply the LDG method to approximating the spatial derivative. We will show the constructed uniform L1/LDG scheme is numerically stable for $\mathcal{Q}^k$ elements on cartesian meshes. Furthermore, by the aid of the so-called Stokes projection (see [26]), we obtain the error estimates in L^2 norm for the velocity, the stress (gradient of velocity), and the pressure.

2. If the solution has weak (or low) regularity at the starting time, adopting the non-uniform L1 scheme is one of the best choices. In this situation, we use the non-uniform L1 scheme in time discretization. We still use the LDG method to approximate the spatial derivative. We prove that the established non-uniform L1/LDG scheme is numerically stable for the velocity and the stress. And we obtain the L^2 optimal error estimate for the velocity and L^2 suboptimal error estimate for the stress.

The remainder of the paper is organized as follows. In Section 2, we present some notations adopted throughout the paper. In Section 3, we build up the uniform L1/LDG and non-uniform L1/LDG schemes for time-fractional Stokes model (3.1). The L^2-stability and error estimates for the fully discrete schemes are presented. Numerical experiments are provided in Section 4, which support the theoretical analysis. Concluding remarks are given in the last section.