Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2022-0007

Vol. **1**, No. 4, pp. 471-502
November 2022

3D Hyperbolic Navier-Stokes Equations in a Thin Strip: Global Well-Posedness and Hydrostatic Limit in Gevrey Space

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Received 28 March 2022; Accepted 13 April 2022

Abstract. We consider a hyperbolic version of three-dimensional anisotropic Navier-Stokes equations in a thin strip and its hydrostatic limit that is a hyperbolic Prandtl type equations. We prove the global-in-time existence and uniqueness for the two systems and the hydrostatic limit when the initial data belong to the Gevrey function space with index 2. The proof is based on a direct energy method by observing the damping effect in the systems.

AMS subject classifications: 35Q30, 76D03, 76D10

Key words: 3D hydrostatic Navier-Stokes equations, global well-posedness, Gevrey class, hydrostatic limit.

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1 Introduction and the main result

There have been extensive studies on the well-posedness of the Prandtl type equations, while most of them are concerned with the local-in-time existence and uniqueness. Compared with the local theory, the global in time property is far from being well investigated. Here, we mention Xin-Zhang's work [51] on global

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weak solutions and some recent papers [1, 23, 36, 41–43, 50] on global analytic or Gevrey solutions. Note the above results are obtained mainly in the two-dimensional setting so that the global well-posedness of the three-dimensional case remains open.

In this paper, we aim to establish global well-posedness theories for some Prandtl type equations in the three-dimensional (3D) setting. Precisely, we will investigate the global-in-time existence and uniqueness of the hyperbolic version of 3D anisotropic Navier-Stokes equations and 3D hydrostatic Navier-Stokes equations. The proof relies on an observation that the vertical diffusion leads to a damping effect and the argument is a direct energy method. Note that this argument does not apply to the classical Prandtl equation because of the lack of Poincaré inequality in the half-space.

The system of hydrostatic Navier-Stokes equations plays an important role in the atmospheric and oceanic sciences and it describes the large scale motion of geophysical flow as a limit of Navier-Stokes equations in a thin domain where the vertical scale is significantly smaller than the horizontal one. By a proper rescaling (cf. [14, 43, 46] for instance and references therein), the 3D anisotropic Navier-Stokes equations in a thin domain read

$$\begin{cases} \big(\partial_t+u^\varepsilon\cdot\partial_x+v^\varepsilon\partial_y-\varepsilon^2\Delta_x-\partial_y^2\big)u^\varepsilon+\partial_x p^\varepsilon=0, & (x,y)\in\mathbb{R}^2\times]0,1[,\\ \varepsilon^2\big(\partial_t+u^\varepsilon\cdot\partial_x+v^\varepsilon\partial_y-\varepsilon^2\Delta_x-\partial_y^2\big)v^\varepsilon+\partial_y p^\varepsilon=0, & (x,y)\in\mathbb{R}^2\times]0,1[,\\ \partial_x\cdot u^\varepsilon+\partial_y v^\varepsilon=0, & (x,y)\in\mathbb{R}^2\times]0,1[, \end{cases} \tag{1.1}$$

where $u^\varepsilon, v^\varepsilon$ stand for tangential and normal components of the velocity field respectively, and the viscosity coefficient is denoted by ε^2. In this paper, the above system is considered with the following no-slip Dirichlet boundary condition:

$$u^\varepsilon|_{y=0,1}=0, \quad v^\varepsilon|_{y=0,1}=0.$$

By letting $\varepsilon\to 0$, the first order approximation yields the following hydrostatic Navier-Stokes equations:

$$\begin{cases} \big(\partial_t+u\cdot\partial_x+v\partial_y-\partial_y^2\big)u+\partial_x p=0, & (x,y)\in\mathbb{R}^2\times]0,1[,\\ \partial_y p=0, & (x,y)\in\mathbb{R}^2\times]0,1[,\\ \partial_x\cdot u+\partial_y v=0, & (x,y)\in\mathbb{R}^2\times]0,1[,\\ u|_{y=0,1}=0,\ \ v|_{y=0,1}=0, & x\in\mathbb{R}^2,\\ u|_{t=0}=u_0^H, & (x,y)\in\mathbb{R}^2\times]0,1[. \end{cases} \tag{1.2}$$

Here, v is a scalar function and $u=(u_1,u_2)$ is vector-valued, standing for the normal and the tangential velocity fields respectively. Compared with the Navier-Stokes equations, there is no time evolution equation for the normal velocity v